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Name and Address  Date
THE PHOTON SPECTRUM FROM POSITRON ANNIHILATION IN FLIGHT

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree of
Master of Science
in the Department of Physics
University of Saskatchewan

by

Russell Virley Elliott

Saskatoon, Saskatchewan.
April, 1964.

256946

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The shape of the photon spectrum due to the annihilation of positrons in flight has been calculated. Multiple scattering and energy losses of the positrons before annihilation have been taken into account in the calculation. It was found that multiple scattering tends to narrow, rather than broaden, the photon spectrum. Calculations are done to show that this result is to be expected.
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CHAPTER 1

INTRODUCTION

Up until quite recently, most of the measurements on photonuclear reaction cross sections have been made using bremsstrahlung radiation produced by a betatron. This radiation has a broad, gently sloping spectrum and so it is difficult to extract from the yield curves the contributions due to photons with a given value of energy. The results of these measurements have raised many new questions about details of the shape of the cross-section vs energy curve, questions that can be answered only by undertaking extremely laborious experiments. On the other hand, much of the needed information could be obtained easily if a nearly monoenergetic, continuously-variable-in-energy photon source were available.

If nearly monoenergetic, high energy positrons are allowed to annihilate in flight in a target of low-\(Z\) material, photons with a relatively narrow energy spread are produced. The resulting photon spectrum is continuously variable over a wide range of energies if the energy of the incident positrons is continuously variable. This annihilation photon spectrum has already been used successfully to measure photoneutron cross sections in several elements (Seward et al (1960), Miller et al (1962), Fultz et al (1962), (1963) and Bramblett et al (1963)).

The positrons will be produced in the University of Saskatchewan Linear Electron Accelerator. This accelerator is a high current machine, consisting of 4 accelerating sections, with a maximum loaded
energy of 103 MeV. Positrons will be produced at the end of the first accelerator section by a cascade of photons and positron-electron pairs initiated in a thick, high-Z target by electrons that have been accelerated in the first section. The positrons thus produced will be accelerated by the remaining accelerator sections. Before striking the annihilation target, positrons in a narrow energy interval will be selected by collimation and magnetic analysis. Details of the positron production and acceleration system for the Saskatchewan accelerator are given by Lobb (1963).

The purpose of this thesis is to examine the feasibility of obtaining annihilation photons and to calculate the shape of the photon spectrum. These calculations take into account the effect, on the results, of energy loss and multiple scattering of the positrons before annihilation. A computer program is given for the numerical evaluation of pertinent integrals. Graphs and tables of the expected spectrum shape for different incident positron energies and target thicknesses are given. The results derived in this thesis are compared with the experimental results of Hatcher et al (1961).

Some work has been done on the shape of the photon spectrum by other people, notably Miller et al (1959), (1960), Jupiter et al (1959), (1961) and Hatcher et al (1961). These papers are mainly concerned with experimental measurements on the spectrum. However Jupiter et al (1959), Miller et al (1960) and Hatcher et al (1961) give a theoretical prediction
of the spectrum shape. They all, however, neglect the effect of multiple scattering of the positrons in their calculations. Jupiter et al (1959) and Miller et al (1960) also neglect the effect of energy loss of the positrons before annihilation.
CHAPTER 2

CALCULATION OF THE ANNIHILATION PHOTON SPECTRUM

2.1 Processes Caused by High Energy Positrons

The purpose of this chapter is the derivation of an expression for the yield of photons from positrons that are annihilated in flight in a target. In the present calculation we do not consider the effect, on the spectrum, of scattering or energy loss of the positrons before annihilation. However these processes are very important to the shape of the spectrum and are considered in later chapters.

We consider positrons in the relativistic energy region from 10 to 100 MeV that have been produced in a target by pair production.

We calculate the spectrum for only those photons that are produced by two-photon annihilation. In this process, an impinging positron is annihilated with a free electron in the target to emit two photons. In the laboratory system, the forward-going photon takes up most of the momentum of the positron which is moving at a speed near that of light. The other photon is emitted in a backward direction with an energy of about 0.25 MeV. The high energy photons have a very narrow angular distribution in the laboratory system about the forward direction due to relativistic effects (Jupiter et al (1959), p. 11).

Before we proceed with the calculation of the annihilation spectrum, some of the processes which can occur when a positron is moving through matter will be considered. We should be aware of these processes since they may have an effect on the shape of the spectrum.
Three-Photon Annihilation

The contribution to the yield of photons due to 3-photon annihilation is negligible since the probability for this process to occur is of the order of $1/137^{1/2}$ compared to that for 2-photon annihilation (see Heitler (1954), p. 272).

One-Photon Annihilation

There is a contribution to the yield due to 1-photon annihilation. In order that this process can occur, the electron with which the positron is annihilated must be bound to the nucleus so that momentum is conserved. The ratio of the rate of 1-photon annihilation compared to 2-photon annihilation in lead reaches a maximum of about 20 per cent for 5 MeV positrons. Since the cross-section for 1-photon annihilation is proportional to $Z^5$, whereas the cross-section for the 2-photon process is proportional to $Z$, we consider the yield due to 1-photon annihilation to be negligible in a low-$Z$ material. Jupiter et al (1959), p. 10 quote a ratio of about $7.3 \times 10^{-4}$ for the annihilation of 10 MeV positrons in beryllium.

Positronium

If a positron is slowed down sufficiently by radiative or ionization energy losses, a positronium atom may be formed. This atom will decay into two or three photons in a short time. We will however, neglect any contribution to the yield from this process.
Bremsstrahlung

The generation of bremsstrahlung radiation is by far the most important process competing with annihilation for the production of photons in the energy range being considered. The photons produced by bremsstrahlung will add a tail to the annihilation peak. This will cancel, in part, the advantages which might have been gained had the narrow annihilation spectrum been available alone. Fortunately, most of the bremsstrahlung photons lie in the low energy region, the high energy tip of the bremsstrahlung spectrum being about 0.76 MeV below the high energy edge of the annihilation spectrum. The peak of the annihilation spectrum lies between these two values. The spectra for both of these processes are shown in Figure 2.1.

![Figure 2.1](image)

Figure 2.1. Comparison of annihilation and bremsstrahlung spectra for 15 MeV positrons incident on a LiH target.
Prior to annihilation positrons also lose energy due to bremsstrahlung. That this effect is negligible is shown in chapter 3.

In order to minimize the contribution from bremsstrahlung, a material of low \( Z \) is used since the probability per atom for photon emission due to bremsstrahlung is proportional to \( Z^2 \), whereas the probability for annihilation in flight is proportional to \( Z \).

The effects on the shape of the spectrum due to bremsstrahlung are discussed in detail in the papers by Jupiter et al, (1959 and 1961).

Positron-Electron Collisions

That interaction of positrons with the electrons in the target by which the positrons lose energy before annihilation will also be considered. Unlike radiation energy loss, this process does have an appreciable effect on the shape of the annihilation spectrum. This effect also will be considered in Chapter 3.

Pair Production

The high energy photons, produced by annihilation, may be lost from the spectrum due to pair production. The cross section for this process is very small for low-\( Z \) materials (see Heitler (1954) p. 260) so we will neglect the effect in our calculations. In general, photon absorption, arising from this and other processes such as Compton scattering, can be corrected for by the use of published absorption tables. (See Heitler (1954), p. 362, for a discussion of photon absorption).
2.2 Consideration of Heitler's Annihilation Cross Section

In order to calculate the annihilation probability, which is really the equation for the photon spectrum, we will use the differential cross section for positron annihilation as given in Heitler (1954), p. 269. Kendall and Deutch (1956) have found good agreement between the experimental and theoretical annihilation probabilities in anthracene for positrons with kinetic energies between 0.77 and 3.3 MeV. A comparison has also been made by Colgate and Gilbert (1953) for positron energies of 50, 100 and 200 MeV. Reasonably good agreement was found. We can thus assume that the theoretical expression for the annihilation cross-section is sufficiently well established to form a basis for the following analysis.

The equation for the differential annihilation cross section per electron in the centre-of-mass co-ordinate system (CM system) is, from Heitler (1954), p. 269:

\[
d\sigma = \frac{2e^4}{4\rho_0 E_0} \left[ \frac{E_0^2 + \rho_0^2 + \rho_0^2 \sin^2 \alpha}{E_0^2 - \rho_0^2 \cos^2 \alpha} - \frac{2E_0 \rho_0 \sin \alpha}{(E_0^2 - \rho_0^2 \cos^2 \alpha)^2} \right],
\]

(2.1)

where \( d\Omega' \) is the element of solid angle in the CM system,

\( \alpha \) is the photon emission angle in the CM system,

\( \varphi \) is an azimuthal angle appearing in \( d\Omega' \), and is shown in Figure 2.2,
$E_0$ is the total energy of the electron or positron in the CM system,

$e$ is the magnitude of the electronic charge,

and $\rho_0$ is $c\times$ the momentum of the positron in the CM system. This quantity is used by Heitler and will be referred to as momentum in this chapter.

This cross-section denotes the probability per incident positron, per electron that one of the emitted photons will be found in $d\Omega'$ at $\alpha$ and $\phi$.

In the CM system, the positron and electron have equal and opposite momenta; i.e.,

$$\vec{p}_+ = -\vec{p}_-,$$

where $\vec{p}_+$ is the positron momentum.

Hence

$$\rho_0 = \vec{p}_+ = -\vec{p}_-$$

defines $\rho_0$ as used in equation (2.1).

The relationship of $\alpha$ and $\phi$ to $\rho_0$ is shown in Figure (2.2).

The element of solid angle $d\Omega'_i$ may be written as

$$d\Omega'_i = \sin \alpha \, d\alpha \, d\phi,$$

where $\phi$ and $\alpha$ have been defined above. Since the annihilation process is azimuthally symmetrical we can integrate over $\phi$ from zero to $2\pi$. The cross-section as defined by Equation (2.1) then becomes the probability for emission of a photon into $d\alpha$ at $\alpha$ in any azimuthal direction, that is, between two cones with half angles equal to
The solid angle element for this definition of cross section is
\[
(2.3) \quad d\Omega' = 2\pi d\alpha' = 2\pi \sin\alpha d\alpha
\]

The differential cross section for positron annihilation as defined by Equation (2.1) is expressed in terms of variables that are defined with respect to the CM co-ordinate system. We wish to transform this equation so that it is expressed in terms of variables that are defined with respect to the laboratory co-ordinate system (Lab system). We note that the numbers of photons entering a solid angle \(d\Omega'\) in the CM system is equal to the number of photons entering the corresponding solid angle \(d\Omega\) in the Lab system; that is,

\[
\frac{d\sigma(E_0, \vec{p}_0)}{d\Omega'} \frac{d\Omega'}{d\Omega} = \frac{d\sigma(E_r, \vec{p}_r)}{d\Omega},
\]

where \(E_0\) and \(\vec{p}_0\) are the energy and momentum in the CM system and \(E_r\) and \(\vec{p}_r\) are the equivalent quantities in the Lab system. Thus to obtain \(d\sigma d\Omega\), we only need to transform \(d\Omega', E_0\) and \(\vec{p}_0\) to the Lab system using standard Lorentz transformations.

This transformation has been performed by Kendall and Deutch (1956) and the result is given in their paper. For our purposes, however, we use the fact that there is a unique relationship between the emitted
photon energy and its angle in the Lab system which allows us to define a cross section in the Lab system in terms of energy only. We note that the cross section (Equation (2.1)) is written in terms of the positron energy and momentum, but these can be shown to be related directly to the emitted photon energy so that the above statement applies. An expression which leaves energy as the only variable is more useful than one which is given in terms of energy, angle and momentum, since we want to have the cross-section expressed in terms of energy in order to plot the spectrum.

2.3 Lorentz Transformation of $E_\gamma, p_\gamma$ and $\alpha$ to the Lab System

We will use the convention that the CM system moves in the positive $x$ direction with respect to the Lab system. In the CM system, the electron has velocity $-\vec{u}_e$, given by

$$\vec{u}_e = \frac{p_e}{E_\gamma}.$$  

In the Lab system, where the electron is at rest, the CM system moves to the right with velocity $\vec{v}_e$. If we define $\vec{s}$ as the velocity of the CM system relative to the Lab system in units of $c$, we obtain

$$\vec{s} = \frac{\vec{v}_e}{c} = \frac{E_\gamma}{E_\gamma}.$$  

The total energy of the positron in the Lab system can be written in terms of its energy and momentum in the CM system:
Now, since \( \rho^2 = \rho_0^2 + \mu^2 \), where \( \mu = mc^2 \), the electron or positron rest mass energy, Equation (2.5) becomes

\[
E_+ = \frac{E_0 + \rho_0^2/E_0}{\sqrt{1 - \frac{\rho_0^2}{E_0^2}}} = \frac{2E_0^2 - \mu^2}{\mu},
\]

and hence

\[
(2.6) \quad E_0^2 = \frac{\mu}{2} (E_+ + \mu).
\]

With the use of (2.4) and (2.6) \( \delta^2 \) may also be defined in terms of

\[
(2.4a) \quad \delta = \rho_0 E_0 = \frac{(E_0^2 - \mu^2)^{1/2}}{4\mu(E_+ + \mu)^{1/2}} = \frac{(E_+ - \mu)^{1/2}}{(E_+ + \mu)^{1/2}}.
\]

Up to this point \( E \), the total positron energy has been measured in some consistent, but arbitrary system of units. We can make the definition

\[
U = \frac{E}{\mu},
\]

where \( U \) is the total incident positron energy measured in units of positron rest mass energy \( (mc^2) \) units. This definition is useful since it results in a simplification of some of the equations to be derived.

Equations (2.6) and (2.4a) then become

\[
(2.6a) \quad U_0^2 = \frac{1}{2} (U_+ + 1),
\]

and
If $\Theta$ is the angle between the direction of the incident positron and if $\Theta$ and $\alpha$ are the equivalent angle in the CM system, then $\Theta$ and $\alpha$ may be related by the Lorentz transformation:

$$
\cos \Theta = \frac{\cos \alpha + \delta}{1 + \delta \cos \alpha}.
$$

This completes the derivation of the transformation equations that will be useful in subsequent work. Of these, Equations (2.4b), (2.6a) and (2.7) are the most useful.

2.4 The Relativistic Doppler Effect

If an electromagnetic wave quantum (photon) is observed in a frame of reference moving with respect to the one in which it was emitted, its frequency, and hence its energy, is different from that in the reference frame in which it was emitted. This change in frequency is called the relativistic Doppler shift.

We wish to find the frequency $\omega$ in the Lab system of a photon emitted with frequency $\omega'$ in the CM system. The equation for this shift is (see Jackson (1962), p. 364)

$$
\omega = \frac{\omega' (1 - \delta^2)^{\frac{1}{2}}}{1 - \delta \cos \Theta},
$$
where $\theta$ is the angle in the Lab system as defined for Equation (2.7).

With the use of

$$\hbar \omega' = 2\pi E'_\gamma \quad \text{and} \quad \hbar \omega = 2\pi E_\gamma,$$

where $E'_\gamma$ and $E_\gamma$ are the photon energies, the equation for the relativistic Doppler shift becomes

$$E'_\gamma = \frac{E_\gamma'(1 - \xi^2)^{\frac{1}{2}}}{1 - \xi \cos \theta}.$$  \hspace{1cm} (2.8)

$E'_\gamma$, which is the energy of one of the photons in the CM system, is equal to the total energy of either the positron or the electron in this system; that is,

$$E'_\gamma = E_0 = \frac{\mu}{(1 - \xi^2)^{\frac{1}{2}}}.$$  \hspace{1cm} (2.8a)

If this result is substituted into (2.8), we obtain

$$\frac{k}{\mu} = \frac{E'_\gamma}{\mu} = \frac{1}{1 - \xi \cos \theta},$$

where $k$ is the photon energy in the Lab system in $mc^2$ units and $\theta$ is the angle at which it is emitted. From this equation we can see that there is a one-to-one correspondence between $\theta$ and $k$. The relationship between $k$, $\theta$ and $p_0$, the incident positron momentum, is shown in Figure 2.3. If $\theta$ is set equal to zero in Equation (2.8a), the
photon energy becomes a maximum which we will denote by

\[ k_0 = \frac{l}{1 - S} \]  

(2.9)

With the substitution from (2.4b) this equation becomes

\[ k_0 = \left[ 1 - \left( \frac{U_+ - l}{U_+ + l} \right)^2 \right]^{-1} \]

(2.9a)

\[ \approx \frac{1 + \frac{l}{2U_+}}{1 + \frac{l}{2U_+} - 1 + \frac{l}{2U_+}} \]

\[ \approx U_+ + \frac{l}{2}, \]

where contributions of order \( l/U_+^2 \) and greater are neglected because \( U_+ > 20 mc^2 \).

2.5 Formulae for the Lorentz Transformation of the Cross Section

Equation (2.8a) can be written in terms of \( \cos \alpha \) by the use of (2.7),

\[ k = \frac{l}{1 - S^2} \left( 1 + S \cos \alpha \right) \]  

(2.10)

Use of this equation will simplify the transformation of Equation (2.1) to the Lab system.

Simplification of the transformation results if we define all photon energies in terms of their differences with respect to \( k_0 \), the maximum photon energy in the Lab system. This is equivalent to shifting the origin
of the spectrum from 0 to \( k_0 \) and interchanging the positive and negative directions. An energy different from \( k_0 \) can be written then as

\[(2.11) \quad \gamma = k_0 - k.\]

Substituting (2.9) and (2.10) into this equation, we obtain

\[(2.11a) \quad \gamma = \frac{S}{1-S^2} \left( 1 - \cos \alpha \right),\]

which may be written as

\[(2.12) \quad \delta \cos \alpha = \delta - \gamma (1 - S^2),\]

and thus

\[(2.13) \quad 1 - \delta^2 \cos^2 \alpha = (1 - \delta^2) \left[ 1 + 2 \delta \gamma - \gamma^2 (1 - S^2) \right].\]

Equation (2.13) will prove to be very useful in transforming the equation for the cross section.

### 2.6 Transformation of the Equation for the Cross Section

Equation (2.1) may be written as

\[
d\sigma = d\Omega \frac{e^s}{4p_0 E_0} \left[ \frac{1 + \frac{p_0^2}{E_0^2} + \frac{p_0^2 \sin^2 \alpha}{E_0^2}}{1 - \frac{p_0^2 \cos^2 \alpha}{E_0^2}} \right]
\]

\[(2.1a) \quad - \frac{2 \left( \frac{p_0}{E_0} \right)^4 \sin^4 \alpha}{\left( 1 - \left( \frac{p_0}{E_0} \right)^2 \cos^2 \alpha \right)^2}.\]

Since \( \delta = p_0 / E_0 \), the expression in the square brackets becomes
\[ \frac{1 + \delta^2 + \delta^2 \sin^2 \alpha}{1 - \delta^2 \cos^2 \alpha} = \frac{2 \delta^4 \sin^4 \alpha}{(1 - \delta^2 \cos^2 \alpha)^2}, \]

which, through the use of the relationship \( \sin^2 \alpha = 1 - \cos^2 \alpha \), becomes

\[ \frac{1 + \delta^2}{1 - \delta^2 \cos^2 \alpha} + \frac{\delta^2 (1 - \cos^2 \alpha)}{1 - \delta^2 \cos^2 \alpha} - \frac{2 \delta^4 (1 - \cos^2 \alpha)^2}{(1 - \delta^2 \cos^2 \alpha)^2}. \]  

(2.14)

Now since we can write

\[ \frac{\delta^2 (1 - \cos^2 \alpha)}{1 - \delta^2 \cos^2 \alpha} = \frac{1 - \frac{\delta^2}{1 - \delta^2 \cos^2 \alpha}}, \]

and

\[ \left( \frac{\delta^2 (1 - \cos^2 \alpha)}{1 - \delta^2 \cos^2 \alpha} \right)^2 = \frac{1 + (1 - \delta^2)^2}{(1 - \delta^2 \cos^2 \alpha)^2} - \frac{2(1 - \delta^2)}{1 - \delta^2 \cos^2 \alpha}, \]

(2.14) becomes

\[ -1 + \frac{2(2 - \delta^2)}{1 - \delta^2 \cos^2 \alpha} = \frac{(1 - \delta^2)^2}{(1 - \delta^2 \cos^2 \alpha)^2}. \]  

(2.15)

Introducing (2.13) into this expression we obtain

\[ -1 + \frac{2}{(1 - \delta^2)[1 + 2 \gamma \delta - \gamma^2 (1 - \delta^2)]} \]

(2.15a)

\[ - \frac{2}{(1 - \delta^2)[1 + 2 \gamma \delta - \gamma^2 (1 - \delta^2)]^2}, \]

which, as we can see, is expressed in terms of variables that are defined relative to the Lab system. Thus we have succeeded in transforming the
expression in the square brackets in Equation (2.1a) to the Lab system.

Expression (2.15a) may be simplified considerably with the use of approximations. With the use of (2.9) we see that

\[(2.16) \quad \frac{2 - \xi^2}{1 - \xi^2} = 1 + \frac{k_0}{2},\]

and since \(\xi \approx 1\) for \(k_0 \geq 20 \text{mc}^2\) we can write

\[1 + \xi \approx 2,\]

and thus

\[(2.16a) \quad \frac{2 - \xi^2}{1 - \xi^2} \approx 1 + \frac{k_0}{2}.\]

Again using the approximation \(\xi \approx 1\) we find that

\[(2.17) \quad 1 + 2\gamma\xi - \gamma^2(1 - \xi^2) \approx 1 + 2\gamma.\]

With the introduction of (2.16a) and (2.17) into (2.15a) we obtain

\[\frac{-1 + 2 + k_0}{1 + 2\gamma} \frac{2}{(1 + 2\gamma)^2},\]

which may be written as

\[(2.15b) \quad \frac{k_0 + 2k_0\gamma - 4\gamma^2}{(1 + 2\gamma)^2} \approx \frac{k_0}{1 + 2\gamma},\]

if \(k_0 + 2k_0\gamma \gg 4\gamma^2\). Note that this approximation is valid for \(\gamma \gg 1\) since \(k_0 \gg 1\). We have thus been able to arrive at the simple approximation given by (2.15b), for (2.15a). The validity of this approximation will be shown later.

In order to complete the transformation of Equation (2.1a) to the
Lab system, we must also transform \( d\Omega'/\rho_0 E_0 \) to the Lab system.

This term appears outside the square brackets in Equation (2.1a). From (2.3) we have

\[
(2.3a) \quad d\Omega = 2\pi \sin \alpha \, d\alpha = -2\pi d(\cos \alpha).
\]

In order to transform \( d(\cos \alpha) \), we use (2.11a),

\[
\cos \alpha = 1 - \frac{\gamma}{\delta} \left( 1 - \delta^2 \right),
\]

so that

\[
(2.3b) \quad 2\pi d(\cos \alpha) = -2\pi \left( \frac{1 - \delta^2}{\delta} \right) d\gamma.
\]

which is the required transformation. Also, with the use of (2.4), (2.6a) and (2.9a) we obtain

\[
\rho_0 E_0 = \frac{\rho_0}{E_0} = \frac{\delta}{2} \mu^2 \left( U_4 + 1 \right)
\]

\[
= \frac{\delta}{2} \mu^2 \left( k_o + \frac{1}{2} \right).
\]

With the substitution of this expression and Equation (2.3b) the term in front of the square brackets in Equation (2.1a) becomes

\[
\frac{\theta^4 d\Omega'}{4 \rho_0 E_0} = \pi \left( \frac{\epsilon^2}{\mu} \right)^2 \left( 1 - \frac{\delta^2}{\delta^2} \right) \frac{d\gamma}{(k_o + \frac{1}{2})},
\]

which may be written as

\[
(2.18) \quad \frac{\theta^4 d\Omega'}{4 \rho_0 E_0} = \pi \left( \frac{\epsilon^2}{\mu} \right)^2 \frac{(2k_o - 1)}{\mu^2 (k_o - 2k_o + 1)(k_o + \frac{1}{2})} \, d\gamma,
\]

because
\[ 1 - \xi^2 = \frac{2k_0 - 1}{k_0^2} \]

and

\[ \xi^2 = \frac{k_0^2 - 2k_e + 1}{k_0^2} \]

This expression is exact in that it involves no approximation. If we multiply it by (2.15a) we obtain an exact expression for the equation of the cross section expressed in terms of variables that are defined relative to the Lab system.

Using approximations similar to those used to derive Equation (2.15b) we can simplify Equation (2.18) to obtain

\[ (2.18a) \quad \frac{e^2}{4\rho_0 E_0} \frac{d\Omega}{\xi} \simeq \frac{n(e^2)^2}{\mu k_0^2} \]

Multiplying this equation by (2.15a) we obtain an approximate expression for the cross section expressed in terms of variables which are defined in the Lab system:

\[ (2.19) \quad d\sigma = 2n(e^2)^2 \frac{1}{\mu k_0 (1 + 2\gamma)} d\gamma \]

This equation can be written in terms of the photon energy \( \kappa \) with the use of

\[ d\gamma = d(k_0 - \kappa) = -dk \]

The minus sign in this equation arises because \( k < k_0 \) and hence \( dk \) is negative. We can change this to a positive quantity if we agree to
change the order of the limits of integration when integrating over $k$.

Thus, Equation (2.19) becomes,

\[
d\sigma = 2\pi \left( \frac{e^2}{\mu} \right)^2 \frac{dk}{k^3 \left[ 1 + 2(k_0 - k) \right]}.
\]

This equation gives the number of photons in the energy range $k$ to $k + dk$ per incident positron per electron.

We have succeeded in transforming the equation for the cross section to the Lab system. Equation (2.20), which is an approximate expression for the cross section in the Lab system, will be the equation used in subsequent work. Any calculations made using the simplified, approximate expression will be much less laborious than if the exact expression had been used.

2.7 Verification of the Validity of the Approximations Used in Deriving Equation (2.19)

In order to show that the approximations involved in deriving equation (2.19) are valid, we can compare it with the exact expression $d\sigma$. The exact expression may be obtained with the use of Equations (2.15a), (2.16) and (2.18):

\[
d\sigma = \pi \left( \frac{e^2}{\mu} \right)^2 \frac{(2k_0 - 1) d\gamma}{(k_0^2 - 2k_0 + 1)(k_0 + \frac{1}{2})} \left[ -1 + \frac{2 + \frac{2k_0}{1 + \delta}}{1 + 2\delta \gamma - \gamma^2(1 - \delta^2)} \right]
\]

(2.21)
The comparison is made in Table 2.1 which shows the ratio of Equation (2.21) to (2.19) for several values of $\gamma$ where $k_z$ is fixed at 30 $mc^2$ units.

Table 2.1
Values of $\frac{d\sigma}{d\omega}$ for various values of $\gamma$
with $k_z = 30 mc^2$ units.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0</th>
<th>0.3</th>
<th>0.6</th>
<th>0.9</th>
<th>1.2</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d\sigma}{d\omega}$</td>
<td>1.0180</td>
<td>1.0402</td>
<td>1.0458</td>
<td>1.0454</td>
<td>1.0423</td>
<td>1.0379</td>
</tr>
</tbody>
</table>

It is evident from Table 2.1 that the approximate equation for the cross section is accurate to within 5 percent for values of $\gamma$ up to at least than 1.5 $mc^2$ units. It will be seen later (Chapter 6) that $\gamma$ seldom exceeds 1.5 $mc^2$ units.

We have shown that the approximate expression is accurate to within 5 percent. Since this error is well within the limits of some of the errors in other parts of the theory we will consider the approximate expression adequate for our purposes.

2.8 Derivation of an Approximation to Equation (2.8a)

Equation (2.8a) gives the relationship between $k_z$ and $\theta$ in the Lab system. This relationship will be used in Chapter 5 when we consider the photon spectrum arising from positrons which are scattered before they are annihilated. However, since the angle in Equation (2.8a) appears in
the argument of a cosine rather than explicitly, it will be necessary to
use an approximation in order to write an algebraic relationship between
$k$ and $\theta$. Equation (2.8a) can be rearranged to yield

\[(2.8b) \quad \frac{L}{k} = 1 - \varepsilon \cos \theta.\]

Since we are dealing with small angles we can make the approximation

\[\cos \theta \approx 1 - \frac{\theta^2}{2},\]

which introduces only a negligible error for angles less than 10 degrees.

With the substitution of this result and Equation (2.9) into (2.8b) we obtain

\[\frac{L}{k} \approx 1 - (1 - \frac{1}{k_o})(1 - \frac{\theta^2}{2})\]

\[\approx \frac{\theta^2}{2} + \frac{L}{k_o},\]

where $\theta^2 / 2 k_o$ has been neglected in comparison to the other terms.

This equation can be rearranged to yield

\[(2.22) \quad \theta^2 \approx 2 \left( \frac{L}{k} - \frac{1}{k_o} \right),\]

which is the required result.

With the use of this equation we can write Equation (2.20) in terms
of the annihilation angle $\theta$ rather than the energy $k$. We obtain

\[\frac{d\sigma}{dk} \frac{dk}{d\theta} = \frac{d\sigma}{dk} \frac{dk}{d\theta} \frac{d\theta}{d\theta} = \frac{d\sigma}{d\theta} d\theta.,\]
in which, by (2.22)

\[(2.23) \quad \frac{dk}{d\theta} = -k^2 \Theta d\theta,\]

where the minus sign which occurs can be eliminated by the proper choice of the order of the limits of integration. Thus

\[(2.24) \quad \frac{d\sigma}{d\theta} = \frac{2\pi (\frac{e}{\mu})^2 (\frac{k}{k_0})^2}{k_0[1 + 2(k_0 - k)]} d\theta,\]

where \(k_0\) is related to the incident positron energy by Equation (2.9a) and \(k = k(\theta)\) is related to \(\Theta\) by Equation (2.22).

2.9 Calculation of the Photon Yield

We wish to calculate the yield of photons that are created with energies between \(k\) and \(k + dk\) in a target of thickness \(dt\). We use Equation (2.20) as a basis for these calculations.

The differential cross section as given by Equation (2.20) has units of area. This corresponds to the area presented by an electron, to an incident positron, effective for creation of a photon with energy between \(k\) and \(k + dk\) or angle between \(\Theta\) and \(\Theta + d\Theta\) if Equation (2.24) is used. In a target presenting an area of \(S\) cm\(^2\) to the incident positrons having thickness \(dt\) cm with \(n\) electrons per cm\(^3\), the number of photons produced with energy \(k\) in \(dk\) per incident positron is given by
\[ dN/dk \, dt = \frac{d\sigma \, dk}{dk} \, n \, S \, dt \, . \]

\( n \), the number of electrons per cm\(^3\), may be written as

\[ n = \frac{N_A \, \rho \, Z}{A} \, , \]

where \( N_A \) is Avogadro's number,

\( \rho \) is the density of the target material,

\( Z \) is the nucleon charge number per molecule, and

\( A \) is the gram-molecular weight.

For a multi-atom molecule Equation (2.26) can be written as

\[ n = N_A \left( \frac{\rho_1 \, Z_1}{A_1} + \frac{\rho_2 \, Z_2}{A_2} + \ldots + \frac{\rho_n \, Z_n}{A_n} \right) \, , \]

where the subscripts refer to the different species of atoms present.

Since

\[ \rho_1 = \frac{\rho \, A_1}{A_1 + A_2 + \ldots + A_n} \, , \]

Equation (2.26a) may be written as

\[ n = N_A \rho \, \frac{\sum_i Z_i}{\sum_i A_i} \, , \]

where \( i \) refers to the different species of atoms present. Substituting this result and Equation (2.20) into Equation (2.25) we obtain

\[ \frac{dN}{dk} \, dt = 2\pi r_0^2 \, N_A \rho \frac{\sum_i Z_i}{\sum_i A_i \, k \, (1 + 2(k_0 - k))} \, dt \, . \]
where \( e^1/\mu = r_e \) is the classical electron radius (Heitler (1954), p. 31).

Equation (2.27) gives the probability of a photon being created with energy between \( k \) and \( k + dk \) per positron incident in a target of thickness \( dt \).

### 2.10 Comparison of Target Materials

It was mentioned in section 2.1 that the probability for the production of bremsstrahlung radiation is proportional to \( Z^2 \), whereas the probability for the production of annihilation radiation is proportional to \( Z \) (see Equation (2.27)). We can compare the relative amount of annihilation radiation to the amount of bremsstrahlung radiation produced in materials of different \( Z \) by using the following rough but simple argument.

The radiation length of a material is defined as the thickness in which, on the average, the energy of a positron is reduced to \( e^{-1} \) times its initial value by bremsstrahlung energy losses. The definition of radiation length is based on the assumption that the incident energy of the positron is high enough so that its range in the material is much greater than the radiation length. Thus for one radiation length in any material equal amounts of energy are lost per positron through the production of bremsstrahlung radiation. Thus the amount of bremsstrahlung radiation produced in one radiation length is the same for all materials.

From Equation (2.27), for a given incident energy we may write
(2.28) \[
\frac{dN}{dk} \Delta \chi = C \left( \sum_i \frac{Z_i}{A_i} \right) \Delta \tau_m,
\]
where
\[
C = 2\pi r_0^2 \frac{N_A}{k_0 \left[ \frac{1}{1 + 2(k_0 - k)} \right]}
\]
is a constant for each material considered and depends only on the incident positron energy and angle of the emitted photon, \(m\) is a subscript that refers to the material under consideration and \(\Delta \tau_m = \rho \Delta t_m\) is the radiation length of the material in \(\text{gm/cm}^2\), \(\Delta t_m\) being the radiation length in cm.

We may use Equation (2.28) to define a quality factor for the production of annihilation radiation in various materials. This quality factor \(Q_m\) denotes the amount of annihilation radiation produced in each material in one radiation length of that material. Thus, \(Q_m\) is a measure of the relative amount of annihilation radiation produced in each material per unit of bremsstrahlung radiation. We have

(2.29) \[
Q_m = \left( \sum_i Z_i / \sum_i A_i \right)_m \Delta \tau_m.
\]

Values of \(Q_m\), \(\Delta \tau_m\) and \(\Delta t_m\) are given for various materials in Table 2.2. The radiation lengths given in this table are from a computer calculation by Beer (1964).
Table 2.2. Tabulation of the quality factor $Q_m$ for various materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>$Q_m$</th>
<th>$\Delta x_m$ (gm/cm$^2$)</th>
<th>$\Delta t$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beryllium</td>
<td>28.3</td>
<td>63.6</td>
<td>34.6</td>
</tr>
<tr>
<td>Lithium Hydride</td>
<td>37.3</td>
<td>75.6</td>
<td>92.2</td>
</tr>
<tr>
<td>Liquid Helium</td>
<td>42.7</td>
<td>85.4</td>
<td>700.3</td>
</tr>
<tr>
<td>Liquid Hydrogen</td>
<td>57.8</td>
<td>57.8</td>
<td>825.8</td>
</tr>
</tbody>
</table>

As may be seen from the table, hydrogen and helium have the highest quality factors. These materials exist in a gaseous state and hence are not in a very dense form, except when liquefied. However, because of the difficulties involved in handling low temperature liquids such as these, we will choose lithium hydride (LiH). It should be noted that beryllium has also been used with some success (see Jupiter et al (1961)). When applied to LiH, Equation (2.27) becomes

\[
\frac{dN}{dk} \frac{dt}{dk} = \frac{0.1503}{k_{*} \left[ 1 + \frac{2}{(k_{*} - k)} \right]} \frac{dk}{dt},
\]

where $d\tau$ is in gm/cm$^2$,

$\rho = 0.82$ gm/cm$^3$ (from Katz (1964)),

$N_{H} = 6.025 \times 10^{23}$ mol/(gm.-mol. wt.),

$\Sigma Z_i = 4$

and $\Sigma A_i = 7.95$ gm/(gm-mol. wt.).

Equation (2.30) is the required equation and will be used in the calculations in Chapter 5 where the effect of the positrons being scattered before annihilation is considered. A similar relationship can be expressed
also in terms of the annihilation angle $\theta$ by the use of Equation (2.24).

2.11 Comparison of the Bremsstrahlung Spectrum with the Annihilation Spectrum

In the preceding section we gave a rough method for calculating the amount of bremsstrahlung radiation produced. This section is concerned with obtaining an equation for the number of bremsstrahlung photons in an energy interval from $k$ to $k + dk$ per positron incident on a target of thickness $dt$.

Since there is not a unique relationship between photon energy and angle of emission for the bremsstrahlung cross section, we have to integrate the bremsstrahlung cross section over the solid angle of acceptance that is used for the annihilation photons. This gives the bremsstrahlung background radiation that we would see when looking at the annihilation spectrum. It turns out, however, that for positrons with high incident energies, the emitted bremsstrahlung radiation is sharply peaked in the forward direction so that we can integrate over all angles and still obtain a good approximation to the number of bremsstrahlung photons being collected in the annihilation photon acceptance cone. The angular distribution of bremsstrahlung
radiation is shown in Figure 2.4. (see Jackson (1963), p. 473). Heitler (1954), p. 247 gives the average half-angle for the angular distribution of bremsstrahlung to be

\[ \bar{\theta} \approx \frac{mc^2}{E_0}, \]

where \( mc^2 \) is the rest mass energy of the positron and \( E_0 \) is the total incident positron energy. This expression is valid for very thin targets where scattering and energy loss of the positrons before bremsstrahlung radiation production are neglected. \( \bar{\theta} \) is about \( 1.7 \times 10^{-2} \) radians for 30 MeV positron and the half-angle of the photon acceptance cone that we have chosen is \( 2.5 \times 10^{-2} \) radians so we see that most of the bremsstrahlung radiation is collected in the photon acceptance cone. From Heitler (1959), p. 249 we obtain an expression for the bremsstrahlung spectrum in the case of complete screening of the nucleus by the atomic electrons:

\[
\phi(k) \frac{dk}{dt} = \frac{2e^2}{137} \frac{N_\alpha}{\rho} \frac{\sum Z_i^2}{\sum A_i} \frac{dk}{k} \frac{(E_\gamma - k)}{E_\gamma} \\
(2.31)
\]

\[
x \left[ \left( \frac{E_\gamma - (E_\gamma - k)}{E_\gamma} - \frac{2}{3} \right) \log \left[ \frac{183 \sum (Z_i)^2}{6} + \frac{2}{3} \right] \right] dt
\]

where \( E_\gamma \) is the incident positron energy in MeV and

\( k \) is the emitted photon energy in MeV.

All of the other quantities appearing in this equation have been defined in preceding sections. This equation is valid for positrons of energies
greater than 10 MeV. Other expressions, valid for lower energies and the case where there is only partial screening, are available and a thorough analysis of this problem would involve the use of these. Koch and Motz (1959) list most of the formulae along with the incident positron and emitted photon energies to which they apply.

The bremsstrahlung radiation spectrum predicted by Equation (2.31) may be compared with the annihilation radiation spectrum predicted by Equation (2.30) for thin targets. This is shown in Figure 2.5.

\[ E_r = 0.76 \]

Figure 2.5 Comparison of bremsstrahlung and annihilation spectra for incident positron energy \( E_r \) greater than 10 MeV.

It should be noted that the maximum energy for the bremsstrahlung spectrum occurs 0.76 MeV below the maximum energy of the annihilation spectrum.
CHAPTER 3

ENERGY LOSSES OF POSITRONS BEFORE ANNIHILATION

3.1 Introduction

The purpose of this chapter is to consider the effects on the photon spectrum of energy losses in the target by the positrons, before annihilation. These energy losses result in the spectrum being broadened over what it would be otherwise.

There are chiefly two ways in which an electron or positron can lose energy in a target.

1. By collision: the incident electron transfers energy to the atom by exciting or ionizing it.

2. By radiation: the incident electron is deflected by the field of the nucleus and in the process emits radiation (bremsstrahlung).

A detailed description of these processes is given in Appendix 1. This chapter will consist of a summary of the results obtained in that appendix.

At energies less than 10 MeV, the average energy loss due to the collision process dominates the average radiative loss which may be neglected. At high energies, between 10 and 100 MeV, the radiative energy losses are, on the average, large compared to the collision losses. The energy at which these losses become equal for a given material depends on and the different species of atoms in the material. In most solids this occurs at an energy in the region from 10 to 200 MeV. Since this is the energy range of interest in the present
investigation we must investigate both of these effects.

The Cerenkov effect is another process through which energy may be lost. These losses are negligible however, in the energy range of interest (Sternheimer (1953)).

Electron energy straggling, which is due to fluctuations in the amount of energy transferred during an interaction, arises in both the collision and radiative processes. This straggling, that gives rise to a distribution in energy of the positrons after they have travelled through part of the target, contributes to the width of the photon spectrum.

The energy straggling in both processes is such that the resulting straggling distribution is not symmetrical in shape and thus the average energy loss does not occur at the peak of the distribution. Relatively rare, high energy loss interactions give rise to a long tail in the distribution curve which causes the lack of symmetry. Low-energy transfers give rise to the peak on the energy straggling distribution curve. This peak defines the most probable energy loss of the positrons which is the energy loss that is pertinent to our calculations. A more detailed discussion of these processes is given in Appendix 1, where it is shown that the most probable collision energy loss has a small but non-zero value in contrast to the most probable radiation energy loss which is zero. The most probable energy loss and the shape of the straggling distributions for both processes are discussed in the next section.
3.2 Energy Loss Calculations

The energy straggling distributions for collision and radiative energy losses are shown in Figure 3.1 for 30 MeV positrons incident on a LiH target that is 2m.m. thick. Several features of the energy loss processes are evident from the figure that is drawn roughly to scale.

![Energy Straggling Distributions](image)

**Figure 3.1.** Comparison of the energy straggling distributions for 30 MeV positrons incident on a LiH target 2.0 m.m. thick. The collision straggling distribution is drawn from curves given by Goldwasser, et al (1952).
We see that the width of the radiation straggling distribution is small compared to the width of the collision straggling distribution and thus we will consider it negligible. Since the total straggling distribution consists of a convolution of these two distributions, we can consider it to be equal, to a satisfactory degree of approximation, to the collision straggling distribution. Goldwasser et al (1952) report that although the average radiation energy loss in gold was twice as great as the average collision energy loss, the radiation straggling distribution was narrow enough so that the shape of the measured total straggling distribution differed only negligibly from that of the collision straggling distribution. For most target thicknesses under consideration, the collision straggling distribution may be considered to be a delta function situated at the most probable energy loss. If the width $\delta E$ becomes significant, we can approximate the distribution by a Gaussian distribution to a satisfactory degree of accuracy. In that case, the effect of this distribution on the width of the photon spectrum can be included by means of a convolution integration of the Gaussian distribution and the incident positron distribution. For our present purposes however, it is sufficient to give the most probable energy loss and full width at half height of the collision straggling distribution.

The most probable energy loss of positrons in LiH is given by
Equation (Al. 8), Appendix 1:

\[ -\Delta E = 763 \times 10^{-2} t \left[ \ln t + 19.40 \right], \]

where \( t \) is the target thickness penetrated in \( \text{gm/cm}^2 \) and is such that \( |\ln t| < 19.40 \),

and \( \Delta E \) is in MeV.

Since \( \ln t \) is a slowly varying function of \( t \), (3.1) can be approximated by

\[ -\Delta E = 1.285t, \]

which is accurate to within a few percent over the range of target thickness of interest, from 0.005 to 0.2 cm.

With the use of this equation, we see that the positron energy defined in Chapter 2 then becomes a function of target thickness:

\[ U(t) = U_\tau - 2.52 t, \]

where \( t \) is in \( \text{gm/cm}^2 \) and \( U \) is in \( \text{mc}^2 \) energy units. With this definition of \( U \), Equation (2.9a) becomes

\[ k_\tau(t) = U_\tau + \frac{1}{2} - 2.52 t, \]

with \( k_\tau(t) \) in \( \text{mc}^2 \) units.

The width of the collision energy straggling distribution is given by Equation (Al. 7), Appendix 1:

\[ \delta E = 3.04 \times 10^{-7} t, \]

where \( \delta E \) is in MeV and \( t \) is in \( \text{gm/cm}^2 \). This equation becomes
(3.4a) \[ \delta E = 5.95 \times 10^{-7} t \]

when \( \delta E \) is expressed in \( mc^2 \) energy units and \( t \) is in \( \text{gm/cm}^2 \).

This completes the list of formulae that will be useful when energy losses of the positrons before annihilation are taken into consideration.
CHAPTER 4
MULTIPLE SCATTERING

4.1 Introduction

The purpose of this chapter is to examine the process of multiple scattering of the positrons before they are annihilated. As seen in Chapter 5, the scattering has an important effect on the shape of the photon spectrum. When a high energy positron enters a target it is scattered, in general, many times by the nuclei in the target before annihilation occurs. We can calculate an expression for the angular distribution of the scattered positrons at a depth $t$ within the target.

We can use the method used by some of the earlier workers in the field, for example, Williams (1939), to explain the process of multiple scattering. We can then indicate the general method used in the more rigorous treatment of Moliere as outlined in the review article by Scott (1963). The results of Moliere's theory show that the multiple scattering distribution can be approximated by a Gaussian curve for purposes of calculation of the effect of scattering on the photon spectrum.

4.2 Comparison of Electron and Positron Single Nuclear Scattering

The theory of single nuclear scattering of electrons forms the basis of multiple scattering calculations. Since we are concerned with the multiple scattering of positrons we must examine the differences between the single scattering theory of electrons and positrons. If these differences are small for the angles and energies under considera-
tion, we can apply the electron multiple scattering results to positrons.

These differences are reviewed by Birkoff (1958), p. 107, 112. He shows that they are important only at large angles for scattering from high-Z materials. Since we are considering a low-Z material (LiH) at small scattering angles, we neglect the differences and apply electron results to positrons. In the discussion that follows, we speak of electron scattering because this is what is discussed in the literature quoted. We must keep in mind, however, that for our purposes the results apply equally well to positrons.

4.3 Description of the Multiple Scattering Process

In this section we will describe multiple scattering using quantitative classical ideas. This method will give us sufficient physical insight into the problem without becoming too burdensome. In the next section, the results of a more exact quantum theoretical treatment are presented. The sources for most of the material in this section can be found in Jackson (1963), p. p. 451-459 and Bethe and Askin (1953), p. p. 283-286.

The multiple scattering distribution arises from a combination of many single scattering distributions. Single scattering is described by the formula for the small-angle Rutherford differential scattering cross section:

\[
\frac{d\sigma}{d\Omega} = \frac{4Z^2e^4}{(\beta \gamma)^2 \Omega^2} \frac{d\Omega}{\Omega^2} = \frac{K}{\Omega^2},
\]
where \( Z \) is the nuclear charge number, 
\( e \) is the electronic charge in Coulombs, 
\( \vec{p} \) is the incident electron's momentum in \( \text{gm. cm/sec.} \), 
\( \vec{v} \) is the incident electron's velocity in \( \text{cm/sec.} \), 
\( \Theta \) is the angle in radians through which the electron is deflected and 
\( d\Omega \) is the element of solid angle.

\( d\Omega \) may be written as

\[
d\Omega = \Theta d\Theta d\phi,
\]

where \( \phi \) is the azimuthal angle. Equation (4.1) represents the probability per nucleus of an impinging electron of velocity \( \vec{v} \) and momentum \( \vec{p} \) being deflected into an element of solid angle \( d\Omega \).

Since the distribution is azimuthally symmetrical, we can integrate over \( \phi \) directly, to obtain the element of solid angle

\[
d\Omega = 2\pi \Theta d\Theta.
\]

Equation (4.1) then becomes

\[
\frac{d\sigma}{d\Omega} d\Omega = \frac{K}{\Theta^4} d\Omega,
\]

which represents the probability per nucleus of an incident electron being scattered into the region between two cones with a one-half apex angle of \( \Theta \) and \( \Theta + \Theta \) respectively.

Because of the \( 1/\Theta^4 \) dependence of the small-angle Rutherford cross section most of the scattering is through very small angles. Jackson (1963), p. 453 shows that at large angles \( (\Theta \sim \pi/2) \) that the small-angle
result is within 30 per cent of the exact Rutherford cross section. Thus, since we are mainly concerned with small angles, we can say that the small-angle approximation is sufficiently accurate for our purposes (see Williams (1939), p. 541, Jackson (1963), p. 453 and Scott (1963), p. 253). With this in mind we introduce the small-angle approximation as defined by Scott (1963), p. 233, which consists of:

(a) replacing \( \sin \theta \) by \( \theta \) and \( \cos \theta \) by \( 1 - \frac{\theta^2}{2} \),

and (b) replacing the upper limit of integration, \( \tau \) by \( \infty \).

The last substitution involves the assumption that the function falls off sufficiently rapidly for large arguments so that it is approximately zero before the angle becomes equal to \( \tau \). This definition of the small-angle approximation will be used in all subsequent work in this chapter.

We define the impact parameter as the distance of closest approach of a passing electron to a nucleus if there is no interaction between the nucleus and the electron. For impact parameters of the order of, and greater than the atomic radius, the field of the nucleus that deflects the incident electron is reduced in magnitude due to the screening effect of the atomic electrons. This screening reduces the angle of deflection of the incident electron. This means that the scattering cross section flattens off at small angles to a finite value, rather than increasing as \( \theta^{-1} \) as shown by Equation (4.1a). Figure 4.1 illustrates this behaviour. The angle at which the Rutherford cross section starts to flatten off due to screening is known as the screening angle and will be
denoted by $\Theta_{\text{min}}$.

Following Jackson (1963), p. 453 we have the expression

$$\frac{d\sigma}{d\Omega} = \frac{K}{(\Theta^2 + \Theta_{\text{min}}^2)^2} \ d\Omega,$$

which takes into account the change in shape of the cross-section-vs-angle curve due to the screening effect. This equation may be integrated over all angles (from 0 to $\infty$ in the small-angle approximation) to obtain the probability per atom for an incident electron to be scattered in any direction:

$$\sigma = \int_0^\infty \frac{d\sigma}{d\Omega} \ d\Omega = \int_0^\infty \frac{K}{(\Theta^2 + \Theta_{\text{min}}^2)^2} \ 2\pi \Theta \ d\Theta$$

$$= \frac{\pi K}{\Theta_{\text{min}}^2}$$

The total number of collisions experienced by an electron in penetrating a target to a depth $t$ is

$$n = N t \sigma = \frac{N t \pi K}{\Theta_{\text{min}}^2}.$$
where \( N \) is the number of atoms per \( \text{cm}^3 \),

\( t \) is the target thickness in \( \text{cm} \),

and \( \sigma \) has been defined above.

An electron in traversing a finite thickness of the target will usually undergo very many small-angle deflections and will emerge at a small angle which is a statistical combination of a large number of small deflections. Large-angle scatterings, on the other hand, are very rare. With this in mind we define an angle \( \theta_{\text{max}} \), such that an electron penetrating the target to a depth \( t \) will experience, on the average, only one deflection through an angle greater than \( \theta_{\text{max}} \). With the use of Equations (4.1b) and (4.2) we have

\[
I = N t K \int_{\theta_{\text{max}}}^{\infty} \frac{2 \pi \theta d\theta}{(\theta^2 + \theta_{\text{min}}^2)^2} = N t K \pi \frac{1}{\theta_{\text{max}}^2 + \theta_{\text{min}}^2},
\]

\[
\approx \frac{N t K \pi}{\theta_{\text{max}}^2},
\]

since \( \theta_{\text{max}}^2 >> \theta_{\text{min}}^2 \). Hence

(4.3) \[ \theta_{\text{max}} \approx \pi N t K. \]

For targets of thickness \( t \) in which multiple scattering occurs, we are able to divide the scattering distribution up into two regions; one with scattering angles greater than \( \theta_{\text{max}} \), which we ascribe to single scattering only, and the other with angles less than \( \theta_{\text{max}} \) in which multiple scattering occurs.

In the region of multiple scattering we are able to calculate the
mean square scattering angle for a single scattering:

\[
\langle \theta^2 \rangle = \frac{\int_{\theta_{\min}}^{\theta_{\max}} \theta \, d\theta^2}{\int_{\theta_{\min}}^{\theta_{\max}} (\theta^2 + \theta_{\min}^2)^2}
\]

(4.4)

\[
\approx 2 \theta_{\min}^2 \ln \left( \frac{\theta_{\max}}{\theta_{\min}} \right)
\]

if \( \theta_{\max} \gg \theta_{\min} \).

Since each of the single scatterings are independent events, the mean square scattering angle of the multiple scattering distribution is equal to the sum of the identical mean square scattering angles of the single scattering distributions. Using (4.2a), which gives the average number of collisions, we obtain the mean square scattering angle for the multiple distribution:

\[
\langle \theta^2 \rangle = n \langle \theta^2 \rangle = NNtk \ln \left( \frac{\theta_{\max}}{\theta_{\min}} \right)^2
\]

(4.5)

which may be written as

\[
\langle \theta^2 \rangle = \theta_{\max}^2 \ln n,
\]

(4.5a)

with the use of (4.3).

Up until now we have said nothing about the shape of the multiple scattering distribution. In order to arrive at an expression for the distribution we shall consider the scattering angles that are projected on planes parallel to the incident electron direction. We shall call these
angles projected angles (after Scott (1963), p. 234). The scattering angles $\Theta$ and $\varphi$ that we have been using, will be called spatial angles to differentiate them from the projected angles. The projected angles, which are denoted by $\alpha$, and $\alpha_2$ are shown in Figure 4.2(a). They are projected on planes which are perpendicular to one another.

![Diagram](image)

Figure 4.2(a) Illustration of the projected angles $\alpha_1$ and $\alpha_2$ compared to the spatial angles $\Theta$ and $\varphi$. (b) This diagram shows the projected and spatial angles represented as vectors in a plane perpendicular to the incident electron direction.

For small angles, negligible error is introduced if the scattering angles are represented as vectors in a plane perpendicular to the incident electron direction. To show this for the projected angle $\alpha_1$, say, we use the relationship

$$\vec{s}_1 = \mathcal{R} \vec{\alpha}_1,$$

where $\vec{s}_1$ is a vector and $\mathcal{R}$ is a radius as shown in Figure 4.2(a).

For $\mathcal{R} = 1$, we see that the angles can be represented as vectors as
shown in Figure 4.2(b). In Figure 4.2(b), \( \alpha \) and \( \alpha_z \) correspond to
cartesian co-ordinates and \( \theta \) and \( \varphi \) correspond to plane polar co-
ordinates.

The statistical distribution of the projected angles \( \alpha \), and \( \alpha_z \) are
identical so we have equal mean square scattering angles:

\[
\langle \alpha^2 \rangle = \langle \alpha_z^2 \rangle.
\]

Since the successive single scatterings are independent events
and they correspond to a finite mean square scattering angle (Equation
(4.4)), we can use the central-limit theorem of statistics (see Cramér
(1946), p. 214) to show that for \( n \) scatterings the resultant projected
scattering distribution is a Gaussian curve symmetrical about \( \alpha = \frac{\pi}{2} \).
Thus for the projected angle \( \alpha \), say, we obtain the distribution

\[
P_\alpha (\alpha) \, d\alpha = \frac{1}{\sqrt{2\pi} \langle \alpha^2 \rangle} \, e^{-\alpha^2 / 2 \langle \alpha^2 \rangle} \, d\alpha.
\]

For small deflection angles (as in Figure 4.3(b)) we can write

\[
\Theta^2 = \alpha^2 + \alpha_z^2,
\]

and since by (4.6) the mean square deflections are equal in each direc-
tion we obtain

\[
\langle \Theta^2 \rangle = 2 \langle \alpha_z^2 \rangle.
\]

The distribution in the spatial angles \( \Theta \) and \( \varphi \) is obtained by multi-
plying the Gaussian distributions in \( \alpha \), and \( \alpha_z \):
\[ P(\alpha_1, \alpha_2) d\alpha_1 d\alpha_2 = \frac{1}{\sqrt{2\pi} \langle \alpha_1^2 \rangle \sqrt{2\pi} \langle \alpha_2^2 \rangle} e^{-\left(\frac{\alpha_1^2}{2\langle \alpha_1^2 \rangle} + \frac{\alpha_2^2}{2\langle \alpha_2^2 \rangle}\right)} d\alpha_1 d\alpha_2 \]

\[ = \frac{1}{\pi \langle \alpha_1^2 \rangle} e^{-\frac{(\alpha_1^2 + \alpha_2^2)}{\langle \alpha_1^2 \rangle}} d\alpha_1 d\alpha_2 \]

\[ = \frac{1}{\pi \langle \Theta^2 \rangle} e^{-\Theta^2/\langle \Theta^2 \rangle} d\alpha_1 d\alpha_2. \]

We can express the Cartesian volume element \( d\alpha_1, d\alpha_2 \) as a volume element \( \Theta d\Theta d\phi \) in plane polar co-ordinates so that the above equation may be written as

\[ P(\Theta) d\Theta d\phi = \frac{1}{\pi \langle \Theta^2 \rangle} e^{-\Theta^2/\langle \Theta^2 \rangle} \Theta d\Theta d\phi. \]

Since the distribution is azimuthally symmetrical we can integrate over \( \phi \) to obtain

\[ P(\Theta) d\Theta = \frac{2}{\langle \Theta^2 \rangle} e^{-\Theta^2/\langle \Theta^2 \rangle} \Theta d\Theta. \]  

This equation is valid for a large number of single scatterings and represents the probability of an electron being scattered between \( \Theta \) and \( \Theta + d\Theta \) in any azimuthal direction, after having penetrated the target to a depth \( t \).

In order for Equation (4.9) to represent the multiple scattering distribution there must be a large number of collisions. If there are only a few scatterings there will be a large tail on the multiple scattering
distribution due to single scattering through large angles. This would be the case for thin foils since, by Equation (4.2), the number of collisions is dependent on the thickness of the foil:

\[(4.2) \quad n = \frac{N E K}{\Theta m} \]

The distribution for a small number of collisions is illustrated in Figure 4.3.

The more exact theory considered in the next section gives the multiple scattering distribution over the whole range. In that case the tail due to single scattering is included in the formula for the distribution. For a large number of scatterings, however, the distribution is very nearly Gaussian in shape.

Methods can be devised to show how close the shape of the distribution is to a Gaussian. These will be discussed in more detail after we consider a more accurate theory for the distribution.

The purpose of this section has been to describe multiple scattering and to consider some of the important effects determining the shape of the distribution.
4.4 More Accurate Theory of Multiple Scattering

In this section we describe briefly how the more accurate theory of multiple scattering is derived. We then show that a distribution Gaussian is a very good approximation to the scattering distribution. We can use this more exact theory to calculate the mean square scattering angle that gives the best fit for the Gaussian distribution.

We use the theory of small-angle multiple scattering derived by Moliere (1948) with the corrections given by Bethe (1953) and Nigam et al (1959). The source for this material is a review paper by Scott (1963). It might be mentioned that Snyder and Scott (1949) also derived a theory mathematically equivalent to that of Moliere.

The distribution is calculated in the following way. Suppose \( f_1(\phi, \tau) \, d\phi \) is the probability for a single scattering through a projected angle \( \phi \), into \( d\phi \), in a thickness \( \tau \) and similarly \( f_2(\phi_2, \tau) \, d\phi_2 \) is the probability for a scattering through a projected angle \( \phi_2 \) etc. Then, if these events are independent, \( F_2(\phi, \tau) \, d\phi \) the probability of scattering through an angle \( \phi = \phi_1 + \phi_2 \) is given by the convolution of the two separate probabilities:

\[
F_2(\phi, \tau) \, d\phi = d\phi \int_{-\infty}^{\infty} f_1(\phi_1, \tau) f_2(\phi - \phi_1, \tau) \, d\phi_1.
\]

By a theorem of Fourier transform theory, we know that the Fourier transform of a convolution integral is equal to the product of the Fourier transforms of the functions being convoluted. If we denote the Fourier
transform of \( \mathcal{F}_z(\phi, \xi) \) by \( \mathcal{F}_z(\xi, \xi) \) the above equation becomes, when transformed:

\[
\mathcal{F}_n(\xi, \xi) = \mathcal{F}_1(\xi, \xi) \mathcal{F}_2(\xi, \xi) \ldots \mathcal{F}_n(\xi, \xi),
\]

where

\[
\mathcal{F}_1(\xi, \xi) = \int_{-\infty}^{\infty} e^{i \xi \phi} f_1(\phi, \xi) d\phi.
\]

This result can be generalized (Scott (1963), p. 236), for \( n \) scatterings to yield

\[
(4.10) \quad \mathcal{F}_n(\xi, \xi) = \mathcal{F}_1(\xi, \xi) \mathcal{F}_2(\xi, \xi) \ldots \mathcal{F}_n(\xi, \xi),
\]

where \( \mathcal{F}_n(\xi, \xi) \) is the Fourier transform of the probability of an electron being scattered between \( \phi \) and \( \phi + d\phi \) after exactly \( n \) scatterings in a target of thickness \( \xi \).

Since we are considering a single electron being multiply scattered, the scattering events must occur consecutively so that the scattering probability denoted by \( f_2(\phi, \xi) d\phi \) occurs at a larger value of \( \xi \) than does the scattering probability denoted by \( f_1(\phi, \xi) d\phi \). If we separate out this \( \xi \) dependence, it can be shown (Scott (1963), p. 236) that Equation (4.10) simplifies to become

\[
(4.10a) \quad \mathcal{F}_n(\xi, \xi) = e^{-\omega_0 \xi} (\hat{\omega}(\xi))^n \xi^n / n!,
\]

where \( \hat{\omega}(\xi) \) is the Fourier transform of the single scattering distribution and \( \omega_0 \xi \) is the probability that one scattering through any angle whatsoever will occur in \( \xi \) (that is, \( \omega_0 \) is the total single scattering cross
section. Since Equation (4.10a) represents the Fourier transform of
the scattering distribution for exactly \( n \) scatterings we can calculate the
complete distribution for any number of scatterings by summing over all
values of \( n \):

\[
\tilde{F}(\xi, t) = \sum_{n=0}^{\infty} \tilde{F}_n(\xi, t) = \sum_{n=0}^{\infty} e^{-w_n t} \left( \frac{\tilde{\omega}(\xi) t}{n!} \right)^n
\]

Thus the Fourier transform of the multiple scattering distribution
can be put in the form of an exponential. If \( \tilde{\omega}(\xi) \) is known, an inverse
Fourier transformation of (4.11) may be performed to give the projected
multiple scattering distribution function. The projected distributions
for both of the projected angles may then be multiplied in a manner
similar to that used for Equation (4.9) to give the spatial multiple scat­
tering distribution.

Moliere (1947) calculated an expression for the single scattering
distribution. His distribution consisted of the classical Rutherford cross
section (Equation 4.1) multiplied by a correction factor. To derive the
correction factor he used the Fermi-Thomas distribution of electrons
in the atom. With this single scattering distribution Moliere was able
to derive an expression for the screening angle. Nigam et al (1959)
were able to show that Moliere made an invalid approximation in the ex­
pansion for the single scattering distribution so that it was correct only
to first order. This result also gave an incorrect expression for the screening angle.

Moliere used his single scattering distribution and screening angle formula in his multiple scattering theory. He proceeded in much the same way that we have indicated in the first part of this section. Due to the incorrect expression for the single scattering distribution and screening angle, Moliere's multiple scattering theory is valid only to first order. However Scott (1963), p. 281, 288, has shown that the Moliere multiple scattering theory is equivalent to the theory of Nigam et al. for highly relativistic electrons incident on a target of low-Z material, provided the screening angle defined by Nigam et al is used.

The Moliere multiple scattering distribution is written in terms of a Gaussian plus an expansion which corrects the distribution for single scattering. From Scott (1963), p. 270, the Moliere multiple scattering distribution in spatial angles is

\[ F(\theta, \varphi) \, d\theta d\varphi = \frac{\theta}{2\pi} d\theta d\varphi \left[ 2 e^{-\theta^2} + \frac{F'(\varphi)}{B} + \frac{F''(\varphi)}{B^2} + \ldots \right]. \]

Since this distribution is azimuthally symmetrical we can integrate over \( \varphi \) directly. The above equation becomes

\[ F(\theta, \varphi) \, d\theta = \theta d\theta \left[ 2 e^{-\theta^2} + \ldots \right]. \]
\[ \vartheta = \frac{\Theta}{\chi \sqrt{1 + \frac{B}{B^2}}} \]

where \( \vartheta \) is the reduced angle defined by

\[ \Theta = \frac{\Theta}{\chi \sqrt{1 + \frac{B}{B^2}}} \]

In Equation (4.13) \( \Theta \) is the spatial angle discussed in the last section, and \( \chi \sqrt{1 + \frac{B}{B^2}} \) is analogous to \( \langle \Theta \rangle \). \( \chi \) is an angle analogous to \( \Theta_{\text{max}} \) defined by Equation (4.3) and has the same form (see Scott (1963), p. 268):

\[ \chi^i = \frac{4\pi e^4 t}{(B \cdot \vartheta)^2} \sum_i N_i Z_i (Z_i + 1) \]

where the subscript \( i \) refers to the different species of atoms in the target and \( Z_i \) has been replaced by \( Z_i + 1 \) to account for scattering from atomic electrons. We recall that \( \Theta_{\text{max}} \) is defined such that for angles greater than \( \Theta_{\text{max}} \), the incident electron, on the average, suffers only one scattering in passing through a target of thickness \( t \). The term \( B \) in Equations (4.12) and (4.13) is an expansion parameter used by Moliere, and is the solution to a transcendental equation. Scott (1963) p. 270 gives an approximate formula for \( B \) that is valid to within 1 per cent over the range of interest:

\[ B = 1.153 + 1.122 \ln \Omega \]

where

\[ \Omega = \frac{\chi^i}{\chi^x} \]
\( \chi \) is the screening angle analogous to \( \Theta \) defined in the last section. Thus \( \phi \) may be interpreted as the average number of collisions the electron suffers in passing through a target of thickness \( t \) and is analogous to \( n \) defined by Equation (4.2). As mentioned before, in order to get the correct results from the Moliere theory we must use the screening angle defined by Nigam et al. These corrected results compare well with the experimental results of Hanson et al (1951). Scott (1963), p. 281 has given a very good approximation for this screening angle. For highly relativistic electrons incident on a low-\( Z \) target it becomes

\[
\chi^2 = (\frac{1}{12} \chi_s)^2 \left( 1.231 + 1.448 \beta^2 \right),
\]

where \( \chi_s = 8.25 \times 10^{-3} \frac{Z^3}{U_0} \)

and \( U_0 \) is the incident electron energy in \( mc^2 \) units.

For energies greater than 10 MeV or so, \( \beta \approx 1 \) and so the equation for \( \chi^2 \) becomes

\[
(4.17) \quad \chi^2 = 2.29 \times 10^{-4} \frac{Z^\frac{3}{2}}{U_0},
\]

The \( \mathcal{F} \) terms appearing in Equation (4.12) are essentially correction terms to account for the single scattering tail of the scattering distribution. These have been evaluated by Moliere (1948) and Bethe (1953), who present them in tabular form for \( n = 1, 2 \). They are also tabulated by Birkhoff (1958), p. 118, in his review article on multiple scattering.

Let us compare the first term, a Gaussian, in Equation (4.12)
with the Gaussian scattering distribution derived in section (4.3) and
given by Equation (4.9). The first term of Equation (4.12) is, with the
use of (4.13):

\[ P_i(\theta, \varepsilon) = e^{-\frac{\theta^2}{\chi_\varepsilon^2}} \frac{2\pi d\theta}{\chi_\varepsilon^2 B} \]

where

\[ \chi_\varepsilon^2 B = \chi_\varepsilon^2 (1.153 + 1.122 \ln \Omega_0) \]

The Gaussian distribution derived in section (4.3) is

\[ P_\varepsilon(\theta, \varepsilon) = e^{-\frac{\theta^2}{\langle \theta^2 \rangle}} \frac{2\pi d\theta}{\langle \theta^2 \rangle} \]

where

\[ \langle \theta^2 \rangle = \Theta_{\max}^2 \ln n \]

Now since \( n \) is analogous to \( \Omega_0 \) and \( \Theta_{\max} \) is analogous to \( \chi_\varepsilon^2 \) we
can see that these equations have the same form. The main difference
is that \( \ln n \) has been replaced by a term of the form \( a + b \ln \Omega_0 \).

As was indicated in the introduction to this chapter, we would
like to approximate the multiple scattering distribution by a Gaussian curve.

To do this, let us see how the Gaussian given by Equation (4.12a) com-
pares with the complete scattering distribution as given by Equation

(4.12). This comparison is shown in Figure 4.4 from which we can
see that the first term approximation is a rather poor fit.

In order to obtain a better fit we may adjust the 1/e width \( \Theta_\varepsilon \)
or angle at which the distribution falls to 1/e of its value at \( \Theta = \Theta_0 \), where
\( e = 2.718 \ldots \).
The $1/e$ width of Equation (4.12a) is $\chi = B^t$. There are two possible ways of obtaining an acceptable fit. One way would be to calculate the $1/e$ width of a Gaussian which is the same height and has the same area as the complete distribution. It turns out however, that a Gaussian distribution using a $1/e$ width calculated in this manner is wider than the Moliere distribution and therefore unacceptable, (see Scott (1963), p. 296).

Another method is to use the $1/e$ width of the complete Moliere distribution for the $1/e$ width of the Gaussian approximation. A Gaussian distribution calculated in this manner gives an acceptable fit to the Moliere distribution over most of the range of angles with the exception of angles where single scattering occurs. A Gaussian approximation which uses a $1/e$ width calculated by the latter method is shown compared to the Moliere distribution in Figure 4.5.

The only place that these distributions differ appreciably is in the single scattering tail, and the difference is very small if, on the average,
a large number of scatterings occur. Thus, it appears that a Gaussian approximation to the Moliere multiple scattering distribution, using the 1/e width of the Moliere distribution, would be the most satisfactory distribution for our purposes.

We would expect that the 1/e width of the Moliere distribution, and hence the Gaussian approximation, to be slightly narrower than $\chi_c \sqrt{B}$ as given in Equation (4.12a). Scott (1963), p. 287, gives $\Theta_\xi$ to be:

$$\Theta_\xi = \chi_c (1.007B - 1.33)^{1/2}$$

which becomes, upon substitution of Equation (4.15),

$$\Theta_\xi = \chi_c (-0.170 + 1.130 \ln \chi_c)$$

(4.18)

Thus, the Gaussian approximation can be written as (see Equation (4.12a)):

$$P(\Theta) d\Theta = e^{-\Theta^2/(\Theta_\xi)^2} \frac{e \Theta d\Theta}{(\Theta_\xi)^2}$$

(4.19)

where $\Theta_\xi$ is given by Equation (4.18).

This equation is the multiple scattering distribution that we will use in
the derivation of annihilation spectrum. It represents the probability of an electron being scattered between $\theta$ and $\theta + d\theta$ in any azimuthal direction after having traversed a thickness $t$ of the foil. We note that

$$\Omega_0 = \Omega_0(t) \quad \text{and} \quad \chi_0 = \chi_0(t).$$

In Section (4.3) we discussed the validity of the Gaussian approximation for a small number of collisions, that is, for small $\Omega_0$. We decided that it differed mainly from the exact distribution in the single scattering tail. This has been discussed also by Scott (1963), p. 303.

We would like to devise a simple method to estimate the error involved in neglecting the single scattering tail of the Moliere distribution. There are several methods available to do this; none of them are very satisfactory, so we will merely indicate the procedure and leave it at that.

Scott (1963), p. 301, discusses a method for determining the angle at which the area of the tail is negligible compared to the rest of the distribution. It might be possible to arrive at a rough value for the error involved in using the Gaussian approximation by this method. Another possibility is to plot the Moliere distribution for small $\Omega_0$ on Gaussian graph paper. Since a Gaussian curve gives a straight line, the difference between the Moliere distribution plotted in this manner and a straight line would give an indication of the error involved.

Even if we did know the exact error involved in neglecting the tail of the multiple scattering distribution, it would be difficult to estimate
the error due to this in the final photon spectrum. The photon spectrum is obtained by folding the multiple scattering distribution (Equation (4.19)) into the photon spectrum that results from unscattered positrons (Equation (2.3d)).

In all of these calculations we have neglected the effect of energy loss on the multiple scattering distribution. For an electron beam penetrating very thick targets, the angular distribution of the emerging electrons depends upon the energy loss of the electrons. These energy loss corrections will be taken into consideration in Chapter 5.

4.5 Calculation of the Multiple Scattering Distribution for Lithium Hydride

We wish to write Equation (4.18) in terms of the properties of the target material, LiH. First we will calculate an expression for

\[ \chi^2 = \frac{4 \pi e^2 t}{(\beta_0 \gamma)^2} \sum_i N_i Z_i (Z_i + 1) \]

The summation term may be written as

\[ \sum_i N_i Z_i (Z_i + 1) = \frac{N_A \rho [Z_i (Z_i + 1) + Z_2 (Z_2 + 1)]}{A_i + A_2} \]

where \( N_A \) is Avagadre's number = 6.025x10^{23} molecules/mole,
\( \rho \) is the density of LiH = 0.82 gm/cm³,
\( A_i \) is the atomic weight of lithium = 6.94 gm/mole,
\( A_2 \) is the atomic weight of hydrogen = 1.01 gm/mole,
\( Z_i \) is the nuclear charge number of lithium = 3,
and \( Z_2 \) is the nuclear charge number of hydrogen = 1.
Thus

\[ \sum_i N_i Z_i(Z_i+1) = 1.06 \times 10^{2\ast} \rho. \]

For highly relativistic electrons,

\[ (\beta \cdot \gamma)^2 \approx U^2 (mc^2)^2, \]

where \( U \) is the incident electron energy in \( mc^2 \) units.

Thus, Equation (4.14) becomes

\[ X_c^2 = \frac{4\pi \rho t}{U^2} r_o^2 \times 1.06 \times 10^{2\ast} \]

\[ = 1.06 \frac{\rho t}{U^2} \]

where \( r_o = \frac{e^2}{mc^2} = 2.818 \times 10^{-13} \) cm is the classical electron radius and \( t \) is the target thickness in cm.

If we use \( t \) to denote the thickness in gm/cm\(^2\), the above equation becomes

\[ (4.14a) \quad X_c^2 = 1.06 \frac{t}{U^2} \]

Equation (4.17) is

\[ (4.17) \quad X_{\infty}^2 = 2.29 \times 10^{-4} \frac{(Z^2)^{1/2}}{U}. \]

Now

\[ Z^2 = Z_1^2 + Z_\infty^2, \]

where \( Z_1 \) and \( Z_\infty \) have been defined for Equation (4.14) above.

Thus

\[ Z^2 = 3^2 + 1^2 = 10 \]
and Equation (4.17) may be expressed in terms of the properties of LiH:

\[(4.17a) \quad \chi^2 = \frac{4.93 \times 10^{-4}}{U^2} \]

Thus, with the use of Equations (4.14a) and (4.17a), Equation (4.16) may be written as

\[(4.16a) \quad \Omega_0 = \frac{\chi^2}{\chi^2} = 2.15 \times 10^3 \tau, \]

where \(\tau\) is in gm/cm\(^2\).

We are now in a position to express Equation (4.18) in terms of \(U\) and \(\tau\) for LiH. Squaring (4.18) we obtain

\[(\theta_0^2) = \chi^2 \left(-0.170 + 1.130 \ln \Omega_0\right), \]

which becomes with the use of Equations (4.14a) and (4.16a),

\[(4.18a) \quad (\theta_0^2) = (\tau/U^2) \left(9.01 + 1.20 \ln \tau\right), \]

where \(\tau\) is in gm/cm\(^2\). If we use the identity

\[(4.20) \quad \tau = \frac{9.01 + 1.20 \ln \tau}{U^2}, \]

the Gaussian approximation to the multiple scattering distribution, Equation (4.18), becomes

\[(4.19a) \quad P(\theta, \tau) d\theta = e^{-\frac{\theta^2}{c^2}} \frac{2\theta}{c} d\theta = e^{-\frac{\theta^2}{c^2}} d\left(\frac{\theta}{c}\right)^2. \]

This is the equation, together with Equation (4.20a), that will be used
in calculations of the effect of multiple scattering on the annihilation photon spectrum.

Using Equation (4.16a) we can calculate the number of collisions that occur in any target thickness. Since a thin target gives a small number of scatterings and hence a large scattering tail, we should calculate the average number of scatterings in the thinnest target segments to be used. For targets of 0.1 m, the average, about 19 scatterings. According to Scott (1963), p. 288, the Moliere theory is valid in this region but there would be a very large single scattering tail. The thickest target will be about 2 m in thickness. This would give rise, on the average, to 380 collisions and in this case the distribution would have a small single scattering tail.

Thus, although a large error is introduced when the Gaussian approximation is used for very thin targets, most of the target thickness will be such that the Gaussian approximation is valid.
CHAPTER 5

DERIVATION OF AN EXPRESSION FOR THE ANNIHILATION PHOTON SPECTRUM

5.1 Introduction

In this chapter we will derive an expression for the annihilation photon spectrum from positrons that have lost energy and have been multiply scattered in the target. In earlier chapters we have derived separate expressions for the probabilities of multiple scattering and annihilation.

In order to calculate the probability for a photon to be created with an energy $\kappa$ in the range $d\kappa$, from a positron that has been multiply scattered, we multiply the separate probabilities for the occurrence of each of these events. This product will be the probability we require only if the positron beam is not reduced in intensity, due to annihilation, as it passes through the target. In that case, the annihilation probability would be a function of the target thickness. However, since the annihilation cross section is small, we will assume that this effect is negligible.

Energy losses of the positrons before annihilation can be included by using the expression (Equation (3.2)) which takes this into account.

In this chapter we will use $\Theta_s$ to denote the scattering angle and $\Theta_a$ to denote the annihilation angle in the aforementioned equations for probabilities.
5.2 Description of the Combined Processes of Scattering and Annihilation

Specifically, we wish to calculate the spectrum of photons that are collected in a small cone about the forward direction of half angle $\Theta_c$, called the photon acceptance cone. Figure 5.1 shows the relationship between $\Theta_c$ and the scattering and annihilation angles.

![Diagram illustrating the relationship between angles when positron annihilation occurs in conjunction with multiple scattering in a target of thickness $T$.](image)

Figure 5.1 Diagram illustrating the relationship between angles when positron annihilation occurs in conjunction with multiple scattering in a target of thickness $T$.

A positron enters the target and is multiply scattered through an angle $\Theta_s$. Annihilation occurs in $dt$ at $t$ with a photon being emitted at an angle $\Theta_a$ in an azimuthal direction $\varphi$. $\Theta_a$ is measured with respect to the scattered positron direction, and is measured with respect to the plane defined by the incident and scattered directions of the positron. The emitted photon then emerges
from the target at an angle $\theta_r$ with respect to the normal to the foil. For certain values of $\theta_\varepsilon$, $\theta_c$ and $\varphi$, the photon will be emitted into the cone of half-angle $\theta_c$.

We note that the angle $\varphi$ mentioned above is the only azimuthal angle that affects the results if the photon is to be collected in the photon acceptance cone. The multiple scattering distribution is azimuthally symmetrical about the incident positron direction and the emerging photon distribution is azimuthally symmetrical about the normal to the target. The distribution of photons from annihilation is symmetrical about the scattered positron direction, but only those that emerge with the angle $\varphi$ in an appropriate range, will be collected in the photon acceptance cone. Of course, it is necessary that $\theta_\varepsilon$ and $\theta_c$ are not too large so that $\theta_\varepsilon$ is greater than $\theta_\varepsilon$.

The probabilities of a photon being collected in the photon acceptance cone are the same for positive and negative values of $\varphi$ with equal magnitude, that is, the pattern is symmetrical about $\varphi = 0$. Because of this symmetry we need only consider half the pattern:

$0 \leq \varphi \leq \pi$. This will simplify calculations since it eliminates a possible source of ambiguity. Thus, we denote the probability of a photon being emitted into $d\varphi$ at $\varphi$ by

$P(\varphi) d\varphi = \frac{1}{17} d\varphi$. 

(5.1)
5.3 Derivation of the Probability for Photon Emission into the Photon Acceptance Cone

In the derivation that is to follow we will use the expressions for the annihilation and multiple scattering probabilities that were derived in Chapters 2 and 4 respectively. We will use the small-angle approximation for the scattering and annihilation angles. The fact that the angles are indeed very small will be seen in subsequent calculations. A consequence of using the small angle approximation is that there is very little increase in the path length, with scattering angle over the distance that the positron travels before annihilation. Thus we can consider the increase in path length to be negligible and use the path length as the target thickness \( t \) in the equations for all scattering angles.

With the use of Equations (2.30) and (2.23) in Chapter 2 we can calculate the probability of a positron being annihilated to create a photon between the angles \( \Theta_a \) and \( \Theta_a + d\Theta_a \) in any azimuthal direction within a thickness \( t \) at a depth \( c/t \) in a LiH target:

\[
N(\Theta_a) d\Theta_a dt = \frac{\gamma}{k_y} \frac{0.15 \sigma \Phi \kappa \varepsilon}{i + 2(k_a - \kappa)} d\Theta_a dt
\]

where \( \gamma \) is in gm/cm\(^2\) and

\[
\varepsilon = \frac{2(k_a - \kappa)}{\kappa \cdot \kappa}
\]

\( \varepsilon \) is the angle between the directions defined by the positron and emitted photon respectively. \( \kappa \) is defined as the energy of the
emerging photon; that is,

$$k_s = k(\varepsilon)$$

when $$\varepsilon_s$$ is equal to zero; and is given by

$$k_s = U_r + \frac{h}{e}$$

from Equation (2.9a), where $$U_r$$ is the incident positron energy. When

the energy loss of the positrons before annihilation is included, $$U_r$$

becomes a function of the target thickness $$t$$ and hence

$$k_s = k_s(t).$$

$$k$$ is the energy an emerging photon has when emitted at an angle $$\theta_s$$

with respect to the positron direction; that is

$$k = k(\theta_s).$$

If the energy loss of positrons before annihilation is taken into considera-

tion then

$$k_s = k_s(\theta_s, t),$$

an equation that is analogous to

$$k_s = k_s(t)$$

The explicit form for $$k$$ and $$k_s$$ as functions of target thickness will be
given later. In the subsequent work we will use $$n_s$$ and $$k$$ to denote

$$k_s(t)$$ and $$k(\theta_s, t)$$ respectively.

Equation (4.19a) gives the probability of a positron being multiply

scattered between $$\theta_s$$ and $$\theta_s + d\theta_s$$ in any azimuthal direction at a

depth $$t$$ in a LiH target:
where $c\tau$ is the mean square scattering angle defined in Chapter 4.

The explicit form of this term will be given later in this chapter. The probability of a positron\/annihilated in $dt$ at a depth $t$ in a target with the emission of a photon into a solid angle $\Theta_e d\Theta_e d\phi$ about the direction given by $\Theta_e$ and $\phi$ after having been scattered through an angle $\Theta_s$ into $d\Theta_s$ in any azimuthal direction, is the product of the statistically independent probabilities of each of these events:

\begin{equation}
P(\Theta_s, \Theta_e, \phi, t) \, d\Theta_e \, d\Theta_s \, d\phi \, dt
\end{equation}

In order to obtain an equation for the energy spectrum we must calculate the probability of a photon being emitted with a given energy $k$ in $dk$. Since $\Theta_e$ is uniquely related to $k$ (Equation 5.3) this is the same as the probability of a photon being emitted into $d\Theta_e$ about the direction $\Theta_e$ and any azimuthal direction. Thus we may replace the probability defined by (5.2) in Equation (5.5) by

\begin{equation}
\frac{dN}{dk} \, dk \, dt = \frac{0.1503}{k_0 \sqrt{1 + 2(k_0 - k)}} \frac{d\Theta_e}{\Theta_e} \, \frac{d\Theta_s}{\Theta_s} \, d\phi \, dt,
\end{equation}

which is Equation (2.30) as given in Chapter 2.

With the use of this equation we can express the probability of a
positron being annihilated instantaneously with the emission of a photon with energy \( k \) in \( dk \) and azimuthal angle \( \varphi \) in \( d\varphi \), after having been scattered through an angle \( \theta_s \) in \( d\theta_s \) in any azimuthal direction as

\[
\rho(k, \theta_s, \varphi, t) \, dk \, d\theta_s \, d\varphi \, dt = \frac{0.1503 \, e^{-\frac{\theta_s^2}{ct}}}{\sqrt{1 + 2(k_0 - k)\frac{1}{k_0}}} \, dk \, d\varphi \, dt \, d\theta_s.
\]

This may be written as

\[
\rho(k, \theta_s, \varphi, t) \, dk \, d\theta_s \, d\varphi \, dt = \frac{dn}{dk} \, dk \, e^{-\frac{\theta_s^2}{ct}} \, d\varphi \, dt \, d\theta_s,
\]

since

\[
\frac{dn}{dk} = \frac{0.1503}{\sqrt{1 + 2(k_0 - k)\frac{1}{k_0}}}
\]

according to Equation (2.5a).

The probability of finding a photon with an energy \( k \) in the range \( dk \) emitted in any direction from a target of thickness \( T \) can be found by integrating over \( \theta_s \), \( \varphi \) and \( t \). However, this probability is not of interest to us; we wish to calculate the probability for emission of the photons into the acceptance cone of half-angle \( \Theta_c \). This involves an integration over suitable ranges of \( \theta_s \) and \( \varphi \), and to do this we must find a relationship between \( \theta_s \), \( \Theta_c \), \( \varphi \) and \( \Theta_s \), so that we can express
the volume element $d\Theta_z d\varphi$ in terms of $d\Theta_z$ and the differential of some other angle. (We note that $\pi d\Theta_e$ is the element of solid angle about the normal to the target.) We can then find the probability of a photon being collected in the photon acceptance cone by integrating $d\Theta_e$ over the photon acceptance angle from $0$ to $\Theta_e$.

The geometrical relationship between $\Theta_a$, $\Theta_z$, $\varphi$ and $\Theta_\gamma$ shown in Figure 5.1 may be seen more easily if it is represented as in Figure 5.2.

Figure 5.2. (a) Diagram illustrating the geometrical relationship between the angles when annihilation occurs in conjunction with scattering. (b) Plane approximation of spherical triangle $ABC$, valid for small angles.

In the small-angle approximation the spherical triangle in Figure 5.2 (a) may be replaced by a plane triangle with sides proportional to the angles $\Theta_a$, $\Theta_z$ and $\Theta_\gamma$. If the radius of the sphere
is taken to be unity, then the equation for the relationship between angle, radius and arc length,
\[ S_{\theta_i} = R \theta_i \]
becomes
\[ S_{\theta_i} = \theta_i \]
and the sides of the triangle are equal to the deflection angles.

In spherical trigonometry (Handbook of Chem. and Physics, 1958, p. 324), the relationship between the angles in the spherical triangle ABC is
\[ \cos \theta_i = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \varphi, \]
and, if we approximate \( \cos \theta \) by \( 1 - \theta/2 \) and \( \sin \theta \) by \( \theta \), we obtain
\[ \theta_i^2 = \theta_2^2 + \theta_3^2 - 2 \theta_2 \theta_3 \cos \varphi. \]
This is the cosine law in plane trigonometry so we can see that in the small-angle approximation correct results can be obtained by approximating the spherical triangle by a plane triangle with sides equal to the angles that the sides of the spherical triangle subtend at the centre of the sphere.

The triangle in Figure 5.2b may be interpreted as in Figure 5.3 as showing the relation between the position vectors of a point \( P \), at the apex of the triangle, relative to the two different origins \( O \) and \( O' \). \( \varphi \) and \( \theta_2 \) represent the plane polar coordinates of the point \( P \) relative to the origin \( O \) and the polar axis \( OO' \). The element of area given
by \( r \, dr \, d\theta \) in the usual notation for plane polar coordinates \((r, \theta)\) is \( \Theta_s \, d\Theta_s \, d\phi \).

This is proportional to \( d\Theta_s \, d\phi \) that appears in Equation (5.5a) and is the area element that we wish to transform.

However, instead of using \( O \) as the origin and \( O'O \) as the polar axis we can choose \( O' \) and \( O'O \) respectively, in which case the area element is \( \Theta_s \, d\Theta_s \, d\psi \). The integral of some function \( F(P) \) of the position of \( P \) over some area \( A \) can be written as

\[
\iint_A F(P) \, dA,
\]

and this can be represented by an explicit expression in terms of either of the two sets of polar coordinates discussed above:

\[
\iint_A F(P) \, dA = \iint_{(\Theta_s), \varphi(\Theta_s)} \Phi(\Theta_s) \, F_1(\Theta_s, \varphi) \, \Theta_s \, d\phi \, d\Theta_s,
\]

\[
= \iint_{(\Theta_s), \varphi(\Theta_s)} \Phi(\Theta_s) \, F_2(\Theta_s, \varphi) \, \Theta_s \, d\psi \, d\Theta_s,
\]

where the limits of integration are determined by the area \( A \) and the particular coordinates involved, and
\[ F(P) = F_i(\Theta, \Phi) = F_i(\Theta, \psi). \]

We see that the integration element \( F_i(\Theta, \Phi) \Theta_\psi d\Theta_\psi d\Phi \) corresponds to, but is not necessarily equal to \( F_i(\Theta, \psi) \Theta_\psi d\Theta_\psi d\psi \). If \( F(P) dA \) represents the probability for an event to occur in the area element \( dA \) about \( P \), then \( F(P) \) is a probability density and \( F_i(\Theta, \Phi) \Theta_\psi d\Theta_\psi d\Phi \) represents the probability of the event to occur in the region \((d\Theta_\psi, d\Phi)\) about \((\Theta, \Phi)\) and \( F_i(\Theta, \psi) \Theta_\psi d\Theta_\psi d\psi \) that in the region \((d\Theta_\psi, d\psi)\) about \((\Theta, \psi)\). We note that there is no need for the two elements \( \Theta_\psi d\Theta_\psi d\Phi \) and \( \Theta, d\Theta_\psi d\psi \) to be equal.

Hence, in order to transform \( P(k, \Theta, \Phi, \xi) \) (Equation (5.5a)) to the corresponding function in the \((\Theta, \psi)\) system we must find a relationship between \((\Theta, \Phi)\) and \((\Theta, \psi)\). Applying the cosine law to the triangle of Figure (5.3), we obtain

\[ \Theta_\psi^2 = \Theta_\xi^2 + \Theta_\xi^2 - 2 \Theta_\xi \Theta_\psi \cos \psi. \]

We substitute this relationship into Equation (5.5a) and introduce the area element \( \Theta, d\Theta_\psi d\psi \) or equivalently \( d\Theta_\psi d\psi \) to obtain

\[ P(k, \Theta, \psi, \xi) d\rho d\Theta_\psi d\psi d\xi dt \]

\[ \text{Equation (5.9)} \]

\[ \text{which is the required equation. This equation represents the probability for emission of a photon with energy } k \text{ in a range } dK \text{ into } c' \Theta_\psi \]
at $\Theta$, and $d\psi$ at $\psi$ from an elemental target thickness $dt$ at $t$.

This equation is not necessarily equal to Equation (5.5a), however when it is integrated over $\Theta$ and $\psi$ it will represent the same probability as (5.5a) integrated over $\Theta_s \phi$, if equivalent limits of integration are chosen for each expression. $\Theta_s$ appearing in (5.9) can be calculated for a given value of $k$ by using (5.3).

Equation (5.9) is in a desirable form for calculation of the number of photons collected in the photon acceptance cone from a target of total thickness $T$. In order to do this we must integrate over $t, \Theta, \psi$. The range of integration of $t$ is from 0 to $T$ and the range of integration for $\Theta$ is from 0 to $\Theta_c$. The range of integration for $\psi$ can be determined from Figure (5.3): for a given value of $\Theta_c$, that is, a given value of $k$ since $\Theta = \Theta_c(k, \psi)$ can take on all values in the range $(0, \pi)$, and so must be integrated over the corresponding range $(0, \pi)$, or equivalently, $(0, \pi)$.

Hence the probability per incident positron for a photon with energy $k$ in the range $dk$ to be collected in a photon acceptance cone of half-angle $\Theta_c$, from a target of thickness $T$, is

$$
\frac{d\gamma}{dk} = \int_{0}^{\pi} \int_{0}^{\Theta_c} \frac{dN}{dk} \frac{1}{ct} e^{-\frac{\Theta_c + \Theta_e}{ct}} \left( \Theta_e \cos \psi \right) c \psi d\psi d\Theta_c dt
$$

(5.10)

where $I_0(x)$ is a modified Bessel function of zero order defined by

$$I_0(x) = J_0(ix),$$

where $J_0(x)$ is a zero order Bessel function of the first kind.

Using (5.11) we may write (5.10) as

$$\frac{dY}{dk} = \frac{\theta}{\kappa} \int_0^\pi \int_{\Theta_1}^{\Theta_2} e^{-\frac{\Theta_1 + \Theta_2}{\kappa}} I_0(2\Theta_1\Theta_2) d\Theta_1 d\Theta_2.$$

From this equation we obtain the probability per incident positron per unit energy interval for a photon with energy $\kappa$ in the range $d\kappa$ to be collected in the photon acceptance cone of half-angle $\Theta_\kappa$.

$$\frac{dY}{dk} = \int_0^\pi \int_{\Theta_1}^{\Theta_2} e^{-\frac{\Theta_1 + \Theta_2}{\kappa}} I_0(2\Theta_1\Theta_2) d\Theta_1 d\Theta_2.$$

In order to obtain the photon spectrum we plot $dY/dk$ versus $\kappa$. If Equation (5.12) is multiplied by the number of photons incident on the target we obtain the photon yield. Thus, since $dY/dk$ is proportional to the photon yield at a given energy, we will refer to (5.12) as the yield.
Equation (5.12) is valid for monoenergetic positrons incident on the target. If the incident positron pulse is not monoenergetic, which will be the case in an experiment, we must take the shape of the positron spectrum into account when calculating the yield using (5.12). This may be done by folding the photon spectrum calculated by Equation (5.12) into the incident positron spectrum. This process will give a photon spectrum which takes into account the shape of the incident positron spectrum.

Energy loss by the positrons prior to annihilation also affects the shape of the spectrum and may be taken into account by making the positron energy a function of the target thickness penetrated by the positrons before annihilation. These calculations will be done in the next section.

5.4 Preparation of $\frac{dy}{dk}$ for Integration

In order to evaluate Equation (5.12) it will be advantageous to take the term

\begin{equation}
\frac{dN(t)}{dk} = \frac{0.1503}{k_0[1 + 2(k - k_0)]},
\end{equation}

where $k_0$ and $k$ are functions of the target thickness $t$, outside the integral sign. This can be done for a very thin target, because $k_0$ is a slowly varying function of $t$. From Chapter 3, Equation (3.3), we have
\[ k_0(t) = U_r + t - 2.52 \bar{t} \]

where \( U_r \) is the initial positron energy in \( mc^2 \) units and \( t \) is the target thickness penetrated in gm/cm\(^2\) (50 mg/cm\(^2\) < \( t \) < 170 mg/cm\(^2\)). In order to take \( dN(t)/dk \) outside the integral sign we choose the arithmetic average for the value of \( t \) in Equation (5.13). If \( T \) is the total target thickness then the average value \( \bar{t} = T/2 \) and Equation (5.13) becomes

\[ k_0(\bar{t}) = U_r + \bar{t} - 2.52 \bar{t} \]

If we have a fairly thick target we can break it up into segments, as shown in Figure 5.4, and add the yield from each segment to obtain the total yield.

\[ \frac{dN(t)}{dk} \]

is then calculated for an average value of \( \bar{t} \) for each segment. Thus for any segment \( j \), \( dN/dk \) is replaced by
\[
\frac{dN_j}{dk} = \frac{dN(\bar{t}_j)}{dk}
\]

where
\[
\bar{t}_j = \frac{T_j + T_{j-1}}{2},
\]

the arithmetic mean of the value of \( t \) at each edge of the segment \( J_j \).

Thus the value of \( k_0(\bar{t}_j) \) in \( \frac{dN_j}{dk} \) would be, with the use of Equation (5.13)

(5.13a) \[ k_0(\bar{t}_j) = k_0(\bar{t}_j) = U_t + \frac{t}{2} - 2.52 \bar{t}_j. \]

Hence, for a target consisting of \( n \) segments we can take \( \frac{dN_j}{dk} \) outside the integral sign in the expression for the yield for each segment and Equation (5.12) may then be written as

(5.12a) \[ \frac{dV}{dk} = \frac{dY_1}{dk} + \frac{dY_2}{dk} + \ldots + \frac{dY_n}{dk}, \]

where

\[
\frac{dY_j}{dk} = \int^{t_j}_{t = t_{j-1}} \exp \left( \frac{\Theta(\bar{t}_j) - 2 \cdot \Theta(\bar{t}_j)}{\Theta(\bar{t}_j) \Theta^2} \right) I \left[ 2 \cdot \Theta(\bar{t}_j) \Theta \right] \frac{dt}{
\]

In order to make the integration easier, \( C \) which is a slowly varying function of \( t \) can be approximated by some average value for each segment and still be quite accurate. According to Equation (4.19) we have

(5.15) \[ C = 9.0 \frac{1 + 1.20 \ln t}{U^2}, \]
where \( \tau \) is the target thickness in gm/cm\(^2\),

such that \( \tau / L \leq 0.01 \)

and \( U = U(\tau) \) is the positron energy in mc\(^2\) units at a depth \( \tau \) within the target.

We can use the average value for \( U \) in the target segment under consideration since \( U \) is a slowly varying function of \( \tau \):

\[
\bar{U}_j = U(\bar{\tau}_j) = U_0 - 2.52 \bar{\tau}_j,
\]

where \( U(\tau) \) is defined by Equation (3.2). Since \( \ln \tau \) is a slowly varying function of \( \tau \), negligible error will be introduced by approximating it by \( \ln \bar{\tau}_j \) for the \( j \)-th target segment. Thus, the value of \( C \) for any target segment \( j \) is

\[
(5.15a) \quad \bar{\tau}_j = 0.01 + 1.20 \frac{\ln \bar{\tau}_j}{(U_j)^2}
\]

\( \bar{\tau}_j \) is then effectively a constant for a given target segment and does not enter into the integration.

With these approximations in mind, we find that in order to calculate \( dV_j / d\kappa \) for a given target segment we calculate \( \kappa \circ (\bar{\tau}_j) \) using Equation (5.13a). We then pick a certain value for \( \kappa \circ - \kappa \) which allows us to calculate \( \kappa \) and hence \( \Theta_\alpha \) with the use of Equation (5.3). We then calculate \( \bar{U}_j \), \( \ln \bar{\tau}_j \), and hence \( \bar{\tau}_j \) for the target segment under consideration. Using these values for \( C \) and \( \Theta_\alpha \) we are now able, in principle, to integrate Equation (5.14) to give us a value for \( dV_j / d\kappa \) for the value of \( \kappa \circ - \kappa \) that we have chosen.
The integral in Equation (5.14) can be simplified further through the use of the following definitions:

\begin{equation}
\hat{z} = \frac{\Theta_a^2 + \Theta_b^2}{\zeta; t}
\end{equation}

and

\begin{equation}
\mu = \frac{2 \Theta_a \Theta_b}{\Theta_a^2 + \Theta_b^2}.
\end{equation}

Hence

\begin{equation}
\mu \hat{z} = \frac{2 \Theta_a \Theta_b}{\zeta; t}.
\end{equation}

We note that $\mu$, which is symmetrical in $\Theta_a$ and $\Theta_b$, reaches a maximum of 1 for $\Theta_a = \Theta_b$ and hence the range of $\mu$ is $0 \leq \mu \leq 1$. Since

\begin{equation}
\hat{z} t = \frac{\Theta_a^2 + \Theta_b^2}{\zeta_j},
\end{equation}

a constant for integration over $t$, then

\begin{equation}
\hat{z} dt + d\hat{z} t = 0
\end{equation}

and

\begin{equation}
\frac{dt}{\hat{z}} = -\frac{d\hat{z}}{\hat{z}}.
\end{equation}

The limits of integration over $\hat{z}$ become

\begin{equation}
\lambda_j = \frac{\Theta_a^2 + \Theta_b^2}{\zeta_j T_j} \quad \text{and} \quad \lambda_{j-} = \frac{\Theta_a^2 + \Theta_b^2}{\zeta_j T_{j-}}.
\end{equation}
where $\lambda_j < \lambda_{j-1}$ because $T_j > T_{j-1}$.

If we integrate from $\lambda_j$ to $\lambda_{j-1}$ we can eliminate the minus sign in (5.19).

With the use of these definitions, Equation (5.14) becomes

\[
\frac{dN_j}{dk} = \frac{dN_j}{\varepsilon_j} \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{1}{Z} I_0(\mu z) \, dz \, d\phi
\]

(5.14a)

\[
= \frac{dN_j}{\varepsilon_j} \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} F(\mu \lambda) \, d\theta
\]

Efforts to integrate analytically Equation (5.14a) have ended in failure. Consultation of standard references on Bessel functions (Watson (1944), Erdelyi (1953) and McLachlan (1934)) do not give formulas for the indefinite integrals involved and so we are forced to use numerical integration. This has been done using an electronic computer and forms the topic of the next chapter.

In this chapter we have succeeded in deriving an expression for the yield of photons from a target of thickness $\mathcal{E}$, being collected in a photon acceptance cone of half angle $\Theta_c$. We were then able to make some approximations to the function which makes numerical integration possible.
CHAPTER 6

COMPUTER PROGRAM FOR EVALUATION OF THE PHOTON YIELD

6.1 Introduction

In the last chapter equations were derived which would, in principle, allow us to plot the yield-vs-energy curve or spectrum for photons which are created by positron annihilation. In order to obtain numerical values for the yield of photons at any given photon energy, certain integrals appearing in the equations have to be evaluated numerically. It is the purpose of this chapter to outline an accurate method for this numerical integration.

A computer program has been written which is capable of evaluating the integrals in question and thus of calculating numerical values for the photon yield at a given energy for any target segment. The spectrum for any segment can then be plotted directly from the computer results.

Details of the computer program complete with operating instructions are given in Appendix 2.

Some of the computer results are plotted in this chapter, but most of the results and calculations are presented in Chapter 8.

6.2 Preparation of the Yield Equation for Numerical Evaluation

From Chapter 5, Equation (5.12a) the equation for the photon yield from a target that has been broken up into \( n \) segments is

\[
\frac{dY}{dk} = \sum_{j=1}^{n} \frac{dY_j}{dk},
\]

(6.1)
where, from Equation (5.14a)

\[
\frac{dy_j}{dk} = \frac{dN_j}{dk} \int_{\theta_{j-1}}^{\theta_j} \int_{\lambda_j z}^{\lambda_{j-1} z} I_0(u \lambda) du \, d\theta
\]

The symbols that appear in this equation have been defined in Chapter 5 and will be given again here when needed.

We consider a positron of energy \( U_\ast \) incident on a target of thickness \( T \). The method by which the target is divided is shown in Figure 6.1. \( \bar{\lambda}_j \), the average thickness for any segment, may be calculated using the relationship

\[
\bar{\lambda}_j = \frac{T_{j-1} + T_j}{2}.
\]

In order to evaluate (6.2) and hence (6.1) we pick a certain value for

\[
\gamma_j = \bar{k}_0 - \bar{k}_j.
\]

Since, from (5.13a), we are able to evaluate \( \bar{k}_0 \) for any segment \( j \), that is,

\[
\bar{k}_0 = k_0(\bar{\lambda}_j) = U_\ast + \frac{1}{\bar{\lambda}_j} - 2.52 \bar{\lambda}_j,
\]
we can calculate the energy $\tilde{k}_j$. We then calculate (6.2) for a sufficient number of values of $\gamma_j$ to define the spectrum for the segment $j$. The spectrum from the target of thickness $\mathcal{T}$ is obtained by adding the contributions from each of the segments.

Thus, in Equation (6.2) we can calculate $dN_j/dk$ (from 5.5b):

\begin{equation}
\frac{dN_j}{dk} = \frac{0.1503}{k_v [1 + 2(k_v - \tilde{k}_j)]}, \tag{6.6}
\end{equation}

and $\tilde{\varepsilon}_j$ from (5.15a):

\begin{equation}
\tilde{\varepsilon}_j = \frac{9.01 + 1.20 \tilde{\varepsilon}_j}{(U_\gamma + 0.52 \tilde{\varepsilon}_j)^2}. \tag{6.7}
\end{equation}

In order to evaluate the integral

\begin{equation}
I_j = \int \int \frac{\tilde{\gamma}_j}{\varepsilon_j} I_\circ (\mu, \varepsilon, z) \, d\varepsilon \, d\Theta_\gamma, \tag{6.8}
\end{equation}

we choose the upper limit of integration over $\Theta_\gamma$ to be $\Theta_c = 2.5 \times 10^{-2}$ radians, for example. This gives a photon acceptance cone with an apex angle of about 3 degrees, which should be a fairly representative value for most experimental setups. The limits of integration in the integral over $\varepsilon$ are, from Equation (5.20):

\begin{equation}
\lambda_j = \frac{\Theta_\varepsilon^2 + \Theta_\gamma^2}{\tilde{\varepsilon}_j \varepsilon_j} \quad \text{and} \quad \lambda_{j-1} = \frac{\Theta_{\varepsilon'}^2 + \Theta_{\gamma'}^2}{\tilde{\varepsilon}_j \varepsilon_{j-1}}, \tag{6.9}
\end{equation}

where $\tilde{\varepsilon}_j$ is calculated using (6.7), $\varepsilon_j$ and $\varepsilon_{j-1}$ are fixed for any segment $j$, $\Theta_\gamma$ is the variable of integration in the outer integral and is constant for integration over $\varepsilon$. 

and

$$\Theta^2 = 2 \left( \frac{\bar{k}_{ij} - \bar{k}}{\bar{k}_{ij} k_i} \right)$$

is a constant for any value of $\bar{k}_{ij} - \bar{k}_i$.

The quantity $\mu$, which appears in the argument of the modified Bessel function in the inner integral, is (from Equation (5.18))

(6.10) $$\mu = \frac{2 \Theta \phi}{\Theta^2 + \phi)^2}.$$

In order to put the integrand of the integral over $z$ in a form which can be calculated using the computer we proceed as follows.

The function $e^{-\frac{\pi}{2}}$ can be evaluated directly by the computer which has a special program to do this. Thus, we only have to arrive at an algebraic expression for $I_0(\mu z)$.

It can be shown that for $\mu z \leq 2.5$, $I_0(\mu z)$ can be approximated by

(6.11) $$I_0(\mu z) \approx 1 + \left( \frac{\mu z}{2} \right)^2 + \frac{1}{4} \left( \frac{\mu z}{2} \right)^4 + \frac{1}{36} \left( \frac{\mu z}{2} \right)^6$$

which is a truncation of the power series which defines $I_0(\mu z)$ (see Bowman (1958), p. 41).

For $2.5 \leq \mu \leq 12.5$ it can be shown that $I_0(\mu z)$ can be represented by the truncated asymptotic expansion:
\[ I_0(uz) \simeq \frac{e^{uz}}{\sqrt{2\pi uz}} \left[ 1 + \frac{1}{8uz} + \frac{9}{128uz^2} \right] \]

\[ (6.12) \]

\[ \simeq \frac{e^{uz}}{\sqrt{2\pi uz}} \left[ 1 + \frac{1}{8uz} + \frac{9}{128uz^2} \right], \]

from Bowman (1958), p. 84.

For \( uz > 12.5 \) it can be shown that \( I_u(uz) \) can be represented by the first term in Equation (6.12):

\[ (6.13) \]

\[ I_u(uz) \simeq \frac{e^{uz}}{\sqrt{2\pi uz}}. \]

Each of these approximations were chosen so that the difference between values of the approximate function and the tabulated values (Jahnke et al (1963), p. 216) is less than 1 per cent.

6.3 Method for Integration of the Yield Using Gauss's Rule

For purposes of numerical evaluation of \( I_u \) given by (6.8), we choose Gauss's numerical integration formula (see Scarborough (1958), p. 145). This formula is easily adapted for use on a computer and gives a high degree of accuracy (see previous reference for a discussion of the accuracy of Gauss's formula compared to other numerical integration formulae).

In Gauss's formula the range of integration is subdivided into unequally spaced intervals symmetrically placed with respect to the midpoint of the range of integration. For our purposes we will choose
Il intervals so that the integrand is calculated at 12 points.

Following Scarborough, in order to evaluate an integral using Gauss's formula, we proceed as follows:

Suppose we wish to evaluate the integral

\[ I = \int_a^b f(x) \, dx. \]  

On changing the variable of integration by the substitution

\[ x = (b-a)u + \frac{a+b}{2}, \]

the limits of integration becomes -1/2 and 1/2. We then have

\[ f(x) = f[(b-a)u + \frac{a+b}{2}] = \phi(u), \text{ say}. \]

Then, since

\[ dx = (b-a) \, du, \]

the integral becomes

\[ I = (b-a) \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi u \, du. \]

Using Gauss's formula we can write this integral as

\[ I = (b-a) \left[ R_1 \phi(u_1) + R_2 \phi(u_2) + \ldots + R_n \phi(u_n) \right] \]

where \( u_1, u_2, \ldots, u_n \) are the intervals of division of the transformed range of integration (-1/2, 1/2).

Thus, using (6.14) we obtain

\[ x_1 = (b-a)u_1 + \frac{a+b}{2}, \quad x_2 = (b-a)u_2 + \frac{a+b}{2}, \quad \text{etc}. \]
for the corresponding values of \( \mathbf{z} \). Values of \( \mu_1, \mu_2, \ldots, \mu_n \), which are zeros of the Legendre polynomial \( \mathcal{P}_n(\mu) \) are tabulated by Scarborough (1958), p. 163 and Lowan et al (1942) for various values of \( n \). In our case, as we have mentioned, we chose \( n \) to be 12. The coefficients \( R_1, R_2, \ldots, R_{12} \) which appear in (6.17), have also been tabulated in the above references.

Thus, we can calculate

\[
\phi(\mu_j) = \int \left[ (b-a) \mu_j + (b+a) \right] \]

for any value of \( \mu_j, (j = 1, \ldots, 12) \) and hence evaluate (6.14) using (6.17).

Using this method we can evaluate Equation (6.8). In order to integrate over \( \Theta \), we calculate \( (\Theta_r) \); using Equation (6.15). We must then integrate over \( z \) using Gauss's rule. This procedure is repeated for each value of \( (\Theta_r) \); and hence we can evaluate the outer integral using Gauss's rule.

In order to obtain a given degree of accuracy for an integral \( I \) we proceed as follows. We evaluate the integral using Gauss's rule. Let us denote the value that we obtain for this integration by \( I \). We then divide the range of integration by 2 so that we have 2 integrals:

\[
I = \int_{a}^{b} f(x) \, dx + \int_{b}^{2b-a} f(x) \, dx.
\]
We then evaluate each of these integrals using Gauss's rule and by adding them we obtain an integral which uses twice the number of intervals and hence should be closer to the actual value of the integral than was $I$. Denoting this integral by $I_2$ we check the relative difference between $I$ and $I_2$,

\begin{equation}
\frac{|I - I_2|}{I_2} \times 100 \equiv \Delta,
\end{equation}

where $\Delta$ is a number, usually 1 or 2 per cent, which specifies the accuracy. If the relationship denoted by (6.18) is not satisfied, the range of integration is divided up into 4 segments, 8 segments, etc. until the relative difference between two consecutive integrations satisfies (6.18). This procedure can easily be done on the computer to obtain any desired accuracy.

A question of convergence of the integrals to their actual value arises and is discussed in Appendix 2 for our case.

### 6.4 Range of Validity of the Integral over $\varphi$

It can be shown that the integral

\[ \lim_{\lambda_j \to 0} \int_{\lambda_j}^{\lambda_{j+1}} e^{-\varphi} I_1(\mu \varphi) d\varphi \]

diverges logarithmically. Since $\varphi_j \neq \infty$, the lower limit

\[ \lambda_j = \frac{\Theta_j - i \Theta_j^2}{\varphi_j \varphi_j} \],
does not go to zero unless \( \Theta_\kappa^\alpha + \Theta_\eta^\alpha \) goes to zero. However, in that case the upper limit

\[
\lambda = \frac{\Theta_\kappa^\alpha + \Theta_\eta^\alpha}{\xi_j r}\n\]

also goes to zero and the integral is zero because we are integrating between equal limits.

A problem arises in the first segment where the integral over \( z \) becomes:

(6.19) \[
F(\mu, \lambda) = \int_{\lambda}^{\infty} e^{-z} I_0(\mu z) \, dz.
\]

In this case the range of integration is infinite and we cannot evaluate this integral by numerical integration unless we can approximate the upper limit by some finite value.

The behavior of the integrand in (6.19) is shown in Figure 6.2 for \( \mu = 0 \) and \( \mu = 1 \).

Since the function drops rapidly to zero with increasing \( z \) we are able to choose a finite upper limit of integration which we denote by \( \lambda_\infty \). In order to calculate \( \lambda_\infty \) we choose \( e^{-z} / z I_0(\mu z) \) to be \( 1 \times 10^{-5} \). A smaller value would have given better accuracy but it would also have caused the computer to take longer in each calculation and so \( 1 \times 10^{-5} \) was chosen as the optimum value for our purposes.

\( \lambda_\infty \) can be plotted against \( \mu \) for this value of \( e^{-z} / z I_0(\mu z) \) to obtain a functional relationship between \( \mu \) and \( \lambda \) that can be used
to calculate $\lambda_{\infty}$ in the computer program. The accuracy of this approximation is discussed in Appendix 2.

6.5 Results from the Computer

The target was divided into 8 segments, each 0.2 m.m. in thickness. The incident positron energy was given as 15 MeV and the accuracy of integration was set at 1 per cent. These data were fed into the computer and the photon yield per positron per unit $mc^2$ energy interval was obtained for enough values of $K_s - K$ to define the spectrum for each segment. The results are shown plotted in Figure 6.3 for 4 of the segments.
Figure 6.3. Plot of photon yield versus energy for 15 MeV (29.36 mc^2 units) positrons incident on a target divided up into segments 0.2 mm. in thickness.
The spectra for each of the segments are shifted by \(0.04 \text{ mc}^2\) units to account for the energy loss of the positrons in the target before annihilation. The energy origin in the diagram corresponds to \(E_0 - E = 0\) for the first segment. This is equivalent to a maximum photon energy equal to 29.84 \(\text{mc}^2\) units. The energy units will be changed to MeV in Chapter 8 where calculations on the results are done.
CHAPTER 7

VERIFICATION OF RESULTS

7.1 Introduction

The purpose of this chapter is to examine qualitatively the different factors leading to the shape of the photon energy spectrum described in Chapter 6.

The photon spectrum for the case when the positrons are scattered before annihilation (scattered spectrum) will be compared with the photon spectrum from unscattered positrons (unscattered spectrum). The reason for the difference between these spectra will be discussed.

The reason for the difference between the spectra from consecutive target segments will be considered.

Finally, it will be shown that the equation for the spectrum from scattered positrons reduces to that from unscattered positrons when the scattering is "turned off." This will serve as a check on the validity of the mathematics used.

7.2 A Useful Method of Representing Scattering and Annihilation

The relationship between the angles when annihilation occurs in conjunction with scattering is shown in Figure 7.1. The processes shown in this figure are described in detail in Chapter 5 Section 5. For purposes of discussion, we can consider the vertices of the cones shown in Figure 7.1 to coincide, since $\tau$ is small.
Figure 7.1. Diagram illustrating the relationship between the angles $\theta_s$, $\theta_\pi$, $\theta_\gamma$, and $\phi$ when scattering occurs in conjunction with annihilation in a target of thickness $T$.

Figure 7.2. (a) Diagram illustrating how scattering and annihilation may be represented as cones with common vertices. (b) Diagram illustrating the projection of the cones on a plane perpendicular to the forward direction of the unscattered positron.
This is illustrated in Figure 7.2 for a photon acceptance cone of half-angle $\theta_c$ and for positrons that are scattered through an angle $\theta_s$ so that they lie on the surface of a cone of half-angle $\theta_s$.

Since the angles are small, the radial distances shown in Figure 7.2(b) are proportional to the angles shown in Figure 7.2(a); that is for $\theta_c$,

$$\theta_c' = R\theta_c.$$  

If $R$ is equal to unity then the radial distances in Figure 7.2(b) represent the angles in Figure 7.2(a); that is,

$$\theta_c' = \theta_c.$$

Figure 7.2(b) will prove to be very useful in this chapter because it allows us to picture azimuthal relationships.

The annihilation angle may be represented in a diagram similar to Figure 7.2 and is shown in Figure 7.3. We suppose that the positron
has been scattered through an angle $\theta_s$ and is travelling in the direction $\theta_a$ when annihilation occurs. Since the scattering is azimuthally symmetrical, $AC$ is any line on the surface of the scattering cone and hence $C$ is any point on the circle of radius $\theta_a$ in Figure 7.3(b).

Upon annihilation, the positron emits a photon at an annihilation angle $\theta_a$ with respect to $AC$. Since there is equal probability of the photon being emitted in any azimuthal direction, we obtain an annihilation cone about $\theta_a$ in Figure 7.3(a) and a circle with centre at $C$ and radius $\theta_a$ in Figure 7.3(b). These diagrams provide us with a convenient method for the representation of the angular and azimuthal relationships depicted in Figure 7.1. In the following work we will use a circle or ring pattern similar to Figure 7.3(b) keeping in mind the relationship of Figure 7.3(b) to 7.3(a).

7.3 Comparison Between the Scattered and Unscattered Spectrum

7.3a The Spectrum from a Thin Target

Figure 7.4 shows the photon spectrum from scattered positrons (scattered spectrum) compared with the photon spectrum from unscattered positrons (unscattered spectrum) for a LiH target 0.2 m\,m. in thickness. These are the spectra of photons being collected in a photon acceptance cone of half-angle $\theta_c = 2.5 \times 10^{-2}$ radians. The scattered spectrum is plotted from the computer results for the first target segment as shown in Figure 6.3 and was calculated from Equation (5.14):
Figure 7.4. Comparison of the scattered spectrum with the unscattered spectrum from 15 MeV (29.36 mc²) positrons incident on a LiH target 0.2 mm. thick.

The unscattered spectrum is calculated from Equation (5.5b):

\[ \frac{dN}{dk} \Delta T = \frac{0.1503 \Delta T}{k_0[1+2(k_0-k)T]} \]

where

\[ \Delta T = \int_0^T dt. \]

and \( T \) is the target thickness equal, in this case, to 0.164 gm/cm².
The unscattered spectrum drops to zero when the annihilation angle $\Theta_c$ becomes equal to the photon acceptance angle $\Theta_e$. Since we have a relationship between photon energy and angle,

\[ \Theta_c = 2 \frac{(E_\gamma - E_e)}{E_\gamma E_e} \]  

we can calculate the value of $E_\gamma - E_e$ when $\Theta_c = \Theta_e$. If we denote this value of $E_\gamma - E_e$ by $E_\gamma - E_e$ where

\[ E_\gamma - E_e = 0.28 \text{ mc}^2 \text{ units for } \Theta_e = 2.5 \times 10^{-2} \] and for

\[ E_\gamma = 29.86 \text{ mc}^2 \text{ units as shown in Figure 7.4.} \]

The results shown in Figure 7.1 are surprising because we would expect multiple scattering to broaden, rather than narrow the width of the spectrum at half amplitude (see Hatcher et al (1961), Jupiter et al (1961).

The reason we would expect the scattered spectrum to be broader is based on the following argument.

7.3 b. Spectrum of Photons Going in the Forward Direction

We consider photons that are emitted in the forward direction only, from monoenergetic positrons. By forward direction we mean the direction of the incident positrons.

If the positrons are not scattered upon entering the target, the photons emitted in the forward direction will be monoenergetic also,
since $\Theta_{\alpha}$ is zero and hence $k = k_{\alpha}$ by (7.3). This spectrum is shown in Figure 7.6.

On the other hand, if the positrons are scattered before annihilation, only those photons emitted at the same angle through which the positrons are scattered, will go in the forward direction; that is we must have $\Theta_{\alpha} = \Theta_{s}$ (see Figure 7.5). Furthermore, the azimuthal angle $\phi$, the angle that a photon makes with the plane defined by the scattered and incident positron, must be zero given $\Theta_{\alpha} = \Theta_{s}$ if the photon is to go in the forward direction. This is illustrated in Figure 7.5.

Thus since $\Theta_{\alpha} = \Theta_{s}$ the forward-going photons will have the same angular distribution in $\Theta_{\alpha}$ as the scattered positrons have in $\Theta_{s}$. Hence, according to (7.3), the photon energy spectrum for the forward-going photons will have a finite energy spread. The forward-going photon spectra from scattered and unscattered positrons are shown in Figure 7.6. The vertical scale for the scattered spectrum has been greatly magnified to allow a comparison to be made.
It will be seen later in this chapter that the scattered spectrum shown in Figure 7.6 can be obtained from Equation (7.1) for the case in which the photon acceptance angle, $\theta_c$ is very small.

Extrapolation of the results shown in Figure 7.3 to apply to a finite photon acceptance angle might lead one erroneously to expect the scattered spectrum to be wider than the unscattered spectrum (see Hatcher et al (1961), Jupiter et al (1961)). Our computer results however, show that the scattered spectrum is in fact narrower. That this result is reasonable will be shown in the next section.

7.4 Argument to Show that the Scattered Spectrum is Narrower than the Unscattered Spectrum

7.4 a Calculation Using $\theta_c = 2.5 \times 10^{-2}$ Radian

In the case where we have a photon acceptance cone with a non-zero half-angle, we must consider a non-zero azimuthal angle also. A more detailed consideration of the relationship between $\theta_s$, $\theta_c$ and $\phi$ (as in Figure 7.1) shows that we should expect the scattered spectrum...
to be narrower at half amplitude than the unscattered spectrum, as shown in Figure 7.4.

The relationship between the angles is shown in Figure 7.7 which is similar to Figure 7.3 and uses the conventions discussed there. The two inner circles $\phi_s$ apart at a radius $\theta_s$ represent the cone formed by the positrons that are scattered between $\theta_s$ and $\theta_s + \phi_s$.

![Diagram](image)

Figure 7.7, Diagram showing relationship between $\theta_s$, $\theta_e$, and $\phi_c$ when $\phi_e = \phi_c$.

The probability of a positron being multiply scattered through an angle $\phi_s$ in $\phi_s$ is

$$P(\phi_s) \phi_s = e^{-\frac{\phi_s}{c \tau}} \frac{2 \phi_s}{c \tau}$$

and thus the probability for scattering through an angle $\phi_s$ per unit scattering angle is (Equation (4.19a))
From (7.4) the maximum number of positrons per unit scattering angle occur at (see Appendix 3)

\[ P(\theta_s) = e^{-\frac{\theta_s^2}{\frac{\sqrt{c^2}}{\ell^2}}} \]

(7.5) \[ (\theta_s)_p = \sqrt{\frac{c^2}{2}} \]

the most probable scattering angle. If \( P(\theta_s) \) is plotted versus \( \theta_s \) for the first target segment we obtain a distribution as shown in Figure 7.8.

Figure 7.8. Diagram showing \( P(\theta_s) \) plotted against \( \theta_s \) for the first target segment.

We will first consider photons of energy \( \hbar \cdot \omega \), the energy at which \( \theta_s = \theta_c \) as given by Equation (7.3a), (see also Figure 7.4).

This gives rise to a circle of radius \( \theta_s = \theta_c \) with the centre in \( \Delta \theta_s \) at \( \Delta \) in Figure 7.7.

Consider first \( \theta_s = 0 \). In this case all of the photons emitted
with angle $\theta_\infty$ will be collected in $\theta_\infty$. We see from Figure 7.8 however, that the number of positrons at this angle is zero. Hence there is a negligible contribution to the photon spectrum at $\kappa = \kappa_\infty$ from unscattered positrons.

If $\Theta_\infty$ is equal to $\Theta_\infty$, then about two-thirds of the photons produced are emitted outside $\Theta_\infty$ as shown by the dashed circle in Figure 7.7. However, the scattering distribution has dropped almost to zero at this angle as shown in Figure 7.8 and thus the contribution to the photon spectrum at $\kappa = \kappa_\infty$ is negligible for photons emitted from positrons scattered through this angle.

In the case where $\Theta_\infty = (\Theta_\infty)_\rho$, we can see that the angles in Figure 7.7 are roughly to scale (see Figure 7.8). As may be seen from Figure 7.7 about one-half of the photons emitted with energy $\kappa_\infty$ from positrons scattered through $\Theta_\infty = (\Theta_\infty)_\rho$ are collected inside $\Theta_\infty$. Thus, for this value of $\Theta_\infty$ about one-half of the photons contribute to the photon spectrum at $\kappa = \kappa_\infty$.

If we assume, for purposes of approximation, all of the positrons to be scattered through $\Theta_\infty = (\Theta_\infty)_\rho$ into $d\Theta_\infty$ we can arrive at a very crude estimate of the height of the scattered spectrum compared to the unscattered spectrum at $\kappa = \kappa_\infty$. From the discussion in the preceding paragraph we see that the height of the scattered spectrum should be roughly one-half the height of the unscattered spectrum at $\kappa = \kappa_\infty$. Since, according to Figure 7.4, the height of the scattered
spectrum as obtained by computer is roughly one-half the height of the unscattered spectrum, we see that our crude estimate is in fact quite close.

The foregoing discussion may be used when \( \theta > \Theta \), in which case \( k - k > k - \lambda \). In this case the unscattered spectrum has dropped to zero but the scattered spectrum has a finite value as shown in Figure 7.4.

We may also use this method to show that the height of the scattered spectrum is lower than the height of the corresponding unscattered spectrum between \( k = k_0 \) and \( k = k_\lambda \).

If we consider photons of energy \( k = k_0 \) we see from Figure 7.4 that the spectra coincide. In this case \( \theta = \Theta' \) is equal to zero so that all of the positrons not scattered outside \( \Theta \) will emit photons that will be collected inside \( \Theta \). Since, according to Figure 7.8, the scattering distribution has dropped almost to zero when \( \Theta = \Theta \) and hence a negligible fraction of the positrons are scattered outside \( \Theta \), nearly all of the photons with energy \( k = k_0 \) will be collected inside \( \Theta \).

For segments deeper within the target, \( c \tau \) is larger and hence the scattering distribution is wider. Thus, for certain values of \( c \tau \), the scattering distribution does not drop to zero at \( \Theta = \Theta \) and hence some of the positrons are scattered outside \( \Theta \). On the basis of the argument presented in the preceding paragraph, we see that the photons
emitted at $k = k_0$ from positrons scattered outside $\mathcal{S}_v$ will not be collected inside $\mathcal{S}_v$ and hence the scattered spectrum will drop below the unscattered spectrum at $k = k_0$. This behavior may be seen in the computer results that are plotted in Figure 6.3 where the height of the spectra at $k = k_0$ for the third and fourth segments is appreciably lower than the height for segment 1.

7.4.b Calculation Using Small Acceptance Angle

If we do a calculation for segment 1 using a value of the photon acceptance angle equal to $\sqrt{\varepsilon T}$ we obtain a spectrum as shown in Figure 7.9.

This figure illustrates the behavior of the scattered spectrum at $k = k_0$ when some of the positrons are scattered outside of the photon acceptance cone.

The spectra plotted in this figure reduce to those shown in Figure 7.6 when the photon acceptance angle approaches zero. Thus we can see that the computer results are consistent with the results obtained there. This shows that we have the proper interpretation of the behavior of the scattered spectrum when the photon acceptance angle approaches zero.
In the foregoing work we have succeeded in showing that we should expect the spectrum from scattered positrons to be narrower at half amplitude than the spectrum from unscattered positrons. This serves as a verification of the computer results.

7.5 Comparison of the Spectra From Successive Segments

In the last section we showed that scattering of the positrons before annihilation tends to narrow the annihilation photon spectrum. In Figure 7.4 we have seen that the scattered spectrum is narrower than the unscattered spectrum but that it has the same general shape. This is in agreement with the result that a small amount of scattering would give a rise to a scattered spectrum only slightly different in shape from the unscattered spectrum. Following this line of reasoning we suspect that the shape of the scattered spectrum would depart further from the shape of the unscattered spectrum as the scattering increases. This conjecture is borne out by Figure 7.10 which shows a comparison between the spectra from the first and second segments. These spectra have been plotted as if they had equal maximum energies for ease of comparison. We see that as the scattering increases, the half-width of the spectrum becomes narrower and the tail becomes more prominent.

This behavior is also shown in Figure 7.11 where the spectra
Figure 7.10. Comparison of spectra for segments 1 and 2 with equal maximum energy. These spectra result from annihilation of 15 MeV positrons in target segments 0.2 mm. in thickness.

For segments 2, 3 and 4 have been plotted. From Figure 7.11 we see that as the peak of the spectrum decreases, the tail of the spectrum becomes more prominent, so that the width of the spectra at half amplitude reaches a minimum and then starts to increase again. In our case this occurs in segment 4. This behavior is shown more clearly in Figure 7.12 where the spectra from segments 4, 5 and 7 have been plotted.
The foregoing results are in agreement with the discussion of the effect of scattering on the shape of the spectrum as given in Section 7.4.

7.6 A Check on the Mathematics

In the foregoing sections of this chapter we have considered some qualitative arguments to show that the spectrum from scattered positrons, obtained using the theory derived in the first part of this thesis, is correct. In this section we will show that the equations for the spectrum from scattered positrons reduce to those for unscattered positrons when the scattering
Figure 7.12. Comparison of the spectra from segments 4, 5 and 7.

is removed or "turned off". This will serve as a useful check on the mathematics used in this problem. We will also show that some of the conclusions arrived at by the previous qualitative arguments may be obtained mathematically.

Physically, the processes of scattering and annihilation cannot be separated, since a target in which a positron is annihilated must also scatter the positron. Mathematically, the equation for the photon spectrum arises from the product of the independent probabilities of scattering and annihilation. It is thus valid mathematically to consider "turning off" the scattering probability while leaving the annihilation probability
untouched.

In order to "turn off" the scattering probability, we must make the effective target thickness, which causes scattering, approach zero. This means letting \( t \) approach zero in the scattering distribution:

\[
P(\theta_s, \epsilon) = e^{-\frac{\theta_s^2}{2 \epsilon^2}} \frac{2 \epsilon}{c t}
\]

Since \( t \) also occurs in the annihilation term, allowing \( t \) to approach zero would also "turn off" the annihilation. We notice that \( c = c(\epsilon) \) occurs whenever \( t \) occurs in the scattering distribution, Equation (7.6), (but not in the annihilation term), so that the product \( c t \to 0 \) when \( t \to 0 \) (if \( c \) remains finite). Thus, we make \( c \to 0 \) rather than \( t \to 0 \) and in this manner we obtain a spectrum for photons created in the annihilation within a target of thickness \( t \), of positrons that have not been scattered in the target. We note that \( c t \) is the 1/e width of the scattering distribution.

The equation for the spectrum from unscattered positrons for any segment \( j \) is

\[
\frac{dN_j}{dk}\Delta T = \frac{dN_j}{dk} \int_{T_{n-1}}^{T_n} dt,
\]

where

\[
\frac{dN_j}{dk} = \frac{0.1503}{[1 + 2(k_{0j} - \tilde{k}_j)] k_{0j}}.
\]

If we include scattering of the positrons before annihilation, equation (7.2a) is modified to become (see Equation (5.14) Chapter 5)
where the symbols have been defined in Chapter 5.

We wish to show that

\[ \lim_{c \to 0} \frac{dY}{dk} = \frac{dN}{dk} \]  

This limit will be proven in what follows, first for photons at the energy \( k = k_0 \), and then in the more general case for any arbitrary energy \( k < k_0 \), where \( k \) is defined by

\[ \Theta_0^2 = 2 \left( \frac{k - k_0}{k k_0} \right). \]

7.6a Reduction of the Scattered Spectrum to the Unscattered Spectrum for Photons Emitted at \( \Theta_0 = 0 \).

If we put \( \Theta_0 = 0 \) in Equation (7.6) we obtain

\[ \frac{dY}{dk} = \frac{1.503}{k_0} \int_{T_f}^{T_i} \int_{\Theta_i}^{\Theta_f} e^{-\frac{\Theta_i^2}{2k}} d\Theta_i dT, \]

where \( I_i(\omega) = 1 \) and \( k_0 - k = 0 \). The order of integration has been changed to facilitate calculation. To obtain Equation (7.7) we show that this equation reduces to

\[ \frac{dN}{dk} = \frac{1.503}{k_0} \Delta T_i. \]

The inner integral may be evaluated to yield:

\[ \frac{dY}{dk} = \frac{1.503}{k_0} \left[ \Delta T_i - \int_{T_{i-1}}^{T_i} e^{-\frac{\Theta^2}{2k}} d\Theta \right]. \]
This is the equation for the number of photons at $k = k_0$ from unscattered positrons, minus an integral which depends on $cT$. Consider the first segment where $\mathcal{T}_{-l} = 0$; and $\mathcal{T}_{l} = \mathcal{T}$, Equation (7.8a) may then be integrated (See Appendix 3, Section 2) to yield

$$\frac{dY_i}{dk} = \frac{0.1503}{k_0} \left[ \mathcal{T}_{l} - \mathcal{E}_{l} \mathcal{E}_{l}^i \mathcal{E}_{c} \mathcal{E}_{c}^i \right],$$


Equation (7.8a) could have been integrated for any arbitrary $\mathcal{T}_{l}$ and $\mathcal{T}_{-l}$; however the extra terms in the expression do not contribute to any clearer understanding of the results.

If $cT >> \Theta_c$, the term in the curly brackets in Equation (7.10) becomes large (but smaller than $T$) since $-\mathcal{E}(-\mathcal{X})$ is large and positive for a small value of $\mathcal{X}$. This corresponds to the case where many of the positrons are scattered out of the photon acceptance angle $\Theta_c$.

That the number of photons from the scattered positrons at $k$, are reduced by this effect is shown mathematically in Equation (7.10) by the fact that the curly bracket is subtracted from $\mathcal{T}_{l}$, the term that corresponds to the unscattered spectrum. This process has been discussed qualitatively in Section 7.3 of this chapter.

In order to consider the case where $c \rightarrow 0$ let us first make $c$ small but finite so that $\Theta_c^2 >> cT$. We can then use the asymptotic expansion for $-\mathcal{E}(-\mathcal{X})$ which holds when $\mathcal{X}$ is large (Jahnke, et al (1960), p. 18):
Substituting this result into Equation (7.10), we obtain

\[- \frac{\theta_s^2}{c} \left\{ \frac{E}{c} \left( 1 - \frac{\theta_s^2}{c \tau_i} \right) \right\}_{\theta_s^2 \gg 1} = - \frac{\theta_s^2}{c} \left\{ \frac{c \tau_i}{\theta_s^2} e^{-\frac{\theta_s^2}{c \tau_i}} \right\}.\]

We can take the limit of this equation as \(c \to 0\), in which case this equation reduces to Equation (7.2b), the equation for the unscattered spectrum.

Equation (7.11) can be evaluated for the first segment to show the effect of scattering through \(\theta_s > \sqrt{c \tau_i}\). We have (see Appendix 2)

\[\theta_s^2 / c \tau_i \approx 10\]

and since

\[2 e^{-10} < c \leq 1,\]

the scattered spectrum is equal to the unscattered spectrum at this point. This is shown in Figure 7.4 in Section 7.4 of this chapter.

We have been able to show that the spectrum from scattered positrons reduces to that for unscattered ones, at \(k = k_0\), when \(c \to 0\).

This serves as a good check on the validity of the mathematics used in this problem. We have also shown that the height of the spectrum at \(k = k_0\) depends on the value of the mean square scattering angle \(c \tau_i\), thus confirming the qualitative results obtained in Section 7.4.

7.6b Reduction of the Scattered Spectrum to the Unscattered Spectrum for Photons Emitted in Any Direction

We wish to show that the spectrum from scattered positrons reduces
to that for unscattered positrons when \( c \rightarrow 0 \) for the case when \( 0 \leq \Theta_0 \leq \Theta_c \).

This corresponds to \( k_c \leq k \leq k_{c_c} \) according to Equation (7.3):

\[
(7.3) \quad \Theta_c^2 = 2(k_c - k) \frac{k}{k_c - k}
\]

\( \Theta_c \) cannot be any greater than \( \Theta_0 \) in the evaluation of this limit (Equation (7.7)) since \( (\Theta_0)_{max} = \Theta_0 \) is the acceptance angle for photons from unscattered positrons; that is, in the unscattered case \( \Theta_0 \) is analogous to \( \Theta_0 \).

In the subsection 7.6 a we have shown that Equation (7.7) holds for the case where \( \Theta_0 = 0 \). If we can show that this limit is valid for \( \Theta_0 = \Theta_c \), we expect that it will be true over the whole range \( 0 \leq \Theta_0 \leq \Theta_c \). If this expectation is valid, then we can prove that the scattered spectrum reduces to the unscattered spectrum when the scattering is "turned off."

We wish to show that

\[
(7.14) \quad \lim_{c \to 0} \int_{\Theta_0/0}^{\Theta_0} \int_{\Theta_0/0}^{\Theta_0} I_0 \left( \frac{2 \Theta_0 \Theta_0'}{c \ell} \right) \frac{c \ell t}{c \ell} d \Theta_0' = \Delta T_0',
\]

(see Equation (7.6)). This limit is very difficult to evaluate exactly. Standard references on Bessel functions (Erdelyi (1953), Watson (1944) and McLachlan (1934)) do not contain formulae for the indefinite integrals involved. However in order to evaluate this limit we can approximate the left hand side and then use standard definite integrals.

Assume that we can change the order of integration so that the
integral over $\Theta_y$ is

$$\int_0^{\Theta_y} e^{-\frac{\Theta_a + \Theta_y}{c t}} I_0 \left( \frac{2 \Theta_a \Theta_y}{c t} \right) d\Theta_y^2.$$  

(7.13)

This integral can be evaluated for $\Theta_a \leq \Theta_c$ if we can replace the upper limit of integration by $\infty$. This is, in fact, permissible providing the integrand in Equation (7.13) becomes sufficiently small at or before $\Theta_y = \Theta_c$. It turns out that this can be done for $\Theta_a \neq 0$ provided $c$ is small (and $c \neq 0$).

To show that the integrand is sufficiently small in the region $\Theta_y = \Theta_c$ so that the above procedure is valid, we proceed as follows:

For $c \ll 2 \Theta_a \Theta_y$, the argument of the modified Bessel function in the integrand becomes large. It can then be replaced by the asymptotic expression (Bowman (1958), p. 84):

$$I_0 \left( \frac{2 \Theta_a \Theta_y}{c t} \right) \sim \frac{e^{\frac{2 \Theta_a \Theta_y}{c t}}}{\sqrt{2 \pi \frac{2 \Theta_a \Theta_y}{c t}}}.$$  

The integrand in Equation (7.13) then becomes:

$$\int_0^{\Theta_y} e^{-\frac{\Theta_a + \Theta_y}{c t}} I_0 \left( \frac{2 \Theta_a \Theta_y}{c t} \right) d\Theta_y^2 \sim \frac{e^{\left( \frac{\Theta_c - \Theta_a}{c t} \right)}}{c t} \left( \frac{c - \tau}{c} \right)^t.$$  

We see that we can make the right hand side of this equation as small as we wish by making $c$ sufficiently small.

We assume $c \ll$ is small, but finite, in which case there is some scattering, as shown in Figure 7.13 (a). This would give rise to a
spectrum as shown in Figure 7.13 (b) where the height of the spectrum varies rapidly near \( k = k_c \), (\( \Theta_\alpha = \Theta_c \)), because of the scattering.

![Diagram](image)

Figure 7.13: (a) Diagram illustrating ring pattern for small but finite \( 1/\epsilon \) scattering angle. (b) Comparison of unscattered spectrum with spectrum obtained for small but finite \( 1/\epsilon \) width.

We must then have \( \Theta_\alpha < \Theta_c \) so we put

\[
(7.15) \quad \Theta_\alpha = \Theta_c \pm \sqrt{c \epsilon}
\]

(Refer to diagram).

We note that \( \sqrt{c \epsilon} \) is \( \sqrt{2} \) times the most probable scattering angle so that there will be very little scattering at \( \Theta > \sqrt{c \epsilon} \) since the scattering distribution (per unit angle) is sharply peaked at \( \Theta = \sqrt{c \epsilon} \). Thus, with the use of the value of \( \Theta_\alpha \) given by Equation (7.15), the spectrum will be well defined for the corresponding value of \( k \). Equation (7.15) is convenient since we can make \( \Theta_\alpha \) as close to \( \Theta_c \) as we wish by taking sufficiently small values of \( \epsilon \).
Upon substitution of Equation (7.15) into Equation (7.14) we obtain

\[(7.14a)\]

\[
\frac{(c_\varepsilon t)^{\frac{3}{2}}}{2 \left( \pi \Theta_\varepsilon (\Theta_\varepsilon - \sqrt{c_\varepsilon t}) \right)^{\frac{3}{2}}} \sim \frac{e^{-1} \sqrt{c_\varepsilon t}}{2 \sqrt{\pi} \Theta_\varepsilon},
\]

where

\[
\Theta_\varepsilon (\Theta_\varepsilon - \sqrt{c_\varepsilon t}) = \Theta_\varepsilon^2 - \sqrt{c_\varepsilon t} \Theta_\gamma \sim \Theta_\varepsilon^2,
\]

for \( \Theta_\varepsilon \gg \sqrt{c_\varepsilon t} \).

Now, as \( c \to 0 \) the right hand side of (7.14a) approaches zero and \( \Theta_\varepsilon \to \Theta_c \) by Equation (7.15). We have assumed that \( c \) is small but finite. Let us make the further condition that \( c \) is small enough so that Equation (7.14a), which is the integrand of Equation (7.13) at \( \Theta_\gamma = \Theta_\varepsilon \), is effectively zero. This will allow us to replace the upper limit of integration in Equation (7.13) by \( \infty \), so that we can replace the indefinite integral by a definite integral. We note that this process is possible only because we have made \( c \) small, so in effect we are evaluating the limit of Equation (7.13) as \( c \to 0 \).

Equation (7.13) can then be written

\[(7.13a)\]

\[
e^{-\frac{\Theta_\gamma^2}{c_\varepsilon t}} \int_{0}^{\infty} e^{-\frac{\Theta_\gamma^2}{c_\varepsilon t}} I_0 \left( \frac{2 \Theta_\gamma \Theta_\varepsilon}{c_\varepsilon t} \right) d\Theta_\gamma.
\]

If we put \( \Theta_\gamma = \mu \) Equation (7.13a) becomes

\[
e^{-\frac{\Theta_\varepsilon^2}{c_\varepsilon t}} \int e^{-\frac{\mu^2}{c_\varepsilon t}} I_0 \left( \frac{2 \Theta_\varepsilon \mu}{c_\varepsilon t} \right) d\mu.
\]

Now
\[
\int_0^\infty e^{-pu} I_0(2atu^\dagger) du = \rho^{-1} e^{a/p}, \quad \Re(e) > 0
\]


If \(\alpha = \Theta \alpha / ct\) and \(\rho = 1 / ct\), then (7.13a) becomes

\[
(7.13b) \quad e^{-\frac{\Theta \alpha^2}{ct}} \left( ct \pm \left(\frac{\Theta \alpha}{ct}\right)^2 ct \right) = ct.
\]

With the substitution of this result into Equation (7.12), we obtain

\[
\lim_{c \to 0} \int_{T_j^-}^{T_j^+} \frac{dt}{ct} ct = \Delta T_j^-,
\]

as required. We have kept \(C\) small but finite so that \(1/ct\) has a finite value. We have thus been able to prove Equation (7.12) for \(\Theta \alpha \leq \Theta_c\).

The same results can be obtained for any arbitrary \(\Theta \alpha\) between \(0\) and \(\Theta_c\).

Suppose \(\Theta \alpha = \frac{\Theta_c}{2}\), then (7.14) becomes

\[
\frac{e^{-\frac{\Theta_c}{ct}} \left(\frac{\Theta_c}{ct}\right)^2 ct^t}{(2\pi \Theta_c)^t}.
\]

If we compare this expression to (7.14a) we see that it approaches zero faster as \(ct \to 0\) than does (7.14a), so that in fact Equation (7.13a) can be shown to be true for any value of \(\Theta \alpha\) for \(0 \leq \Theta \alpha \leq \Theta_c\).

We have thus been able to show that the equation for the spectrum from scattered positrons as derived in (Chapter 5) can be reduced to the
equation for the spectrum from unscattered positrons when the scatter-
ing is "turned off". This result provides a check on the correctness of
the theory. It does not serve however, to check the validity of the
assumptions on which the theory is based.
CHAPTER 8

RESULTS AND CALCULATIONS

8.1 Introduction

The purpose of this chapter is to present the results obtained by a typical computer run. For this we have chosen an incident positron energy of 15 MeV and have divided a target that is 1.6 m m. in thickness into 8 segments. When results are required for another energy and target thickness, a computer run can be made and the data analyzed in a manner similar to that in this chapter. Thus, this chapter will serve mainly as an indication of the type of results to be expected.

We will assume that the incident positrons are monoenergetic since the actual energy spectrum of the positron pulse is not known. When an accurate spectrum of the positron pulse becomes available, the photon spectrum from it can be obtained by convolution of the monoenergetic photon spectrum with the positron spectrum.

We will assume the collision energy straggling distribution to be a delta function centered on the most probable energy loss. To show that this assumption is valid we consider the width of the straggling distribution given by Equation (3.4a):

\[ \delta E = 5.95 \times 10^{-1} \xi, \]

where \( \xi \) is in gm/cm\(^2\)

and \( \delta E \) is in mc\(^2\) units.

For a target of 0.2 m.m. in thickness we obtain \( \delta E = 0.0098 \) mc\(^2\) units.
Figure 8.1. Graphs showing photon energy spectra for annihilation of 15 MeV positrons.
If this energy spread is compared to the width of the spectrum from the first target segment in Figure 6.3, we see that it is negligible by comparison and the energy straggling distribution can be considered to be a delta function.

In this chapter the energy scale will be changed from $\text{mc}$ units to $\text{MeV}$.

![Graph](image)

**Figure 8.2.** Half-width in MeV of photon spectrum from 15 MeV monoenergetic positrons versus target thickness.
8.2 Presentation of Data

The spectra from the individual target segments as shown in Figure 6.3 can be added to give the total spectrum for various target segments. In our case we have chosen 8 segments, so that we can obtain the spectrum from a target of any thickness that is a multiple of 0.2 m.m., up to a total target thickness of 1.6 m.m. These results are given in Appendix 2 as computer output data. If we wish to increase the target thickness to 1.8 m.m., the spectrum for the last segment can be calculated by the computer without having to calculate the first 8 segments over again.

![Diagram illustrating the photon conversion efficiency versus target thickness.](image)

Figure 8.3. Diagram illustrating the photon conversion efficiency versus target thickness.
Figure 8.1 shows the spectra for target thickness of 0.8 and 1.6 m.m. The narrower spectrum, which is from the thinner target, is calculated by adding the individual spectra in Figure 6.3. The wider spectrum is calculated in the same manner.

If the spectra for target thickness of 0.4, 0.8, 1.2 and 1.6 m.m. are plotted as in Figure 4.4 we can measure the width of each at half-height (half-width) and obtain a graph of half-width versus target thickness. This is shown in Figure 8.2.

We can calculate the area under the curves for the spectrum to obtain the positron to photon conversion efficiency for targets of various

![Figure 8.4. Diagram illustrating the photon conversion efficiency versus spectrum half-width.](http://example.com/figure84.png)
Figure 8.5. Comparison of spectra from a LiH target 0.8 mm. in thickness for 40 MeV and 15 MeV positrons.
thicknesses. The number of photons per positron is plotted versus target thickness in Figure 8.3 to give an indication of the variation in conversion efficiency with target thickness.

Figure 8.4 shows the conversion efficiency plotted versus the half-width of the photon spectrum.

If we consider, for example, an average beam current in the Saskatchewan Accelerator of 0.2 MA and an electron-to-positron conversion efficiency of the order of $1 \times 10^{-5}$ positrons per electron (Lobb (1964)), we obtain a photon yield of about $1.2 \times 10^6$ photons per sec. This is for a target of thickness 1.0 m, with 15 MeV positrons incident on it. The half-width of the photon spectrum is about 0.16 MeV. This would be broadened considerably when the spectrum of incident positron pulse is taken into account.

A computer run was also made for 40 MeV positrons for a target similar to the one described above. The spectra from a segment 0.8 m, in thickness are compared for the 15 MeV and 40 MeV positrons in Figure 8.5. They are plotted on the same energy scale (that is, $\kappa_0 - \kappa$ for each) for ease of comparison. As may be seen from the figure, the 40 MeV spectrum is considerably broader than the 15 MeV spectrum and would thus be less effective as a tool for measuring photo-nuclear cross sections.

There are no experimental results of sufficient accuracy to check these calculations. Hatcher et al (1961) have measured the photon spec-
trum from a LiH target 1.5 m.m. in thickness for 15 MeV positrons. They obtain a half-width of about 0.45 MeV. The spectrum of their incident positron pulse is about 0.45 MeV in width so that the photon spectrum from monoenergetic positron must be quite narrow.

Thus in order to say anything definite about the shape of the monoenergetic photon spectrum we must use an accurately known positron spectrum with a half-width equal to or less than the half-width of the monoenergetic photon spectrum.
CHAPTER 9

SUMMARY AND DISCUSSION

In the preceding chapters we have shown how we can obtain the photon spectrum from positron annihilation in a target in which energy loss and multiple scattering of the positrons can also occur. The resulting photon spectrum is narrow in energy spread (0.2 MeV for most target thicknesses) and is thus useful for photonuclear work.

As was mentioned in Chapter 8, there are at present no experimental results of sufficient accuracy with which to check the shape of the spectrum as predicted in this thesis. Hatcher et al (1961) have measured the photon spectrum by elastically scattering photons from the 15.11 MeV level in carbon. Higher resolution could be achieved through the use of lithium drift detectors (Tavendale and Ewan (1963)). The usefulness of this method has been investigated at the University of Saskatchewan by Lobb (1964) and it shows considerable promise.

Also in order to check the shape of the spectrum calculated in this thesis, the shape of the incident positron spectrum must be well known. This follows because the final photon spectrum consists of a convolution of the photon spectrum from monoenergetic positrons with the incident positron spectrum. In addition, consideration of this convolution process shows that the half-widths of contributing spectra must be approximately equal if the resulting spectrum is to have maximum intensity for a given half-width.

In the calculations in this thesis, we have considered the incident
positron beam to have zero cross-sectional area. In practice, however, the positron beam will have a small, but non-zero cross-sectional area. How much effect this will have on the photon spectrum is not known. However we do know that it will have some effect since the photon acceptance angle is not the same for positrons at the centre of the beam as for positrons at the edge. This is illustrated in Figure 9.1 (b). The effect will probably result in a somewhat wider spectrum, than that from a beam of zero cross-sectional area.

Figure 9.1. Diagram illustrating the arrangement of target and detector for different beam configurations.
A similar effect arises when the detector is not placed symmetrically on the axis as shown in Figure 9.1 (c). Again, although the exact change in the shape of the photon spectrum would be difficult to predict, we can say that the spectrum will be broadened since the width of the spectrum is a strong function of the photon acceptance angle.

In Chapter 8, Figure 8.5, we have shown the spectra from a target 0.8 m.m. in thickness for 15 and 40 MeV positrons. Calculations should be made for energies between these values so that the photon yield versus incident energy can be plotted.
APPENDIX 1

ENERGY LOSSES OF THE POSITRONS BEFORE ANNIHILATION

In this Appendix we will consider the details of positron collision and radiation energy losses before annihilation. This will serve as a reference to chapter 3 where the results of this appendix are used.

Al.1 Collision Energy Loss

The theory of energy losses by collision of electrons in matter has been investigated quite thoroughly. (see review articles by Bethe and Ashkin (1953), Price (1955) and Birkhoff(1958). Essentially, the process occurs by a transfer of energy to the atomic electrons, raising them to excited energy levels including the continuum. The Möller cross section for scattering from atomic electrons is used as the basis for our calculation.

It is useful to compare the energy loss for positrons with that for electrons in the hope that the differences will be small. If this is true the formulae derived for electron energy loss can then be used for positrons. Rohrlich and Carlson (1954) have made these comparisons for the energy range of interest (10-100 MeV) and they find that the differences between the formulae are less than 3 per cent, which is negligible for our purposes.

For reference, we give the formula for the average energy loss of relativistic electrons, (see Sternheimer (1953));
\[ - \frac{dE}{dx} = \frac{2\pi e^*}{m v^2} \frac{N_a \rho Z}{A} \left\{ \ln\left(\frac{m v^2 E}{2I^2(1-\beta^2)}\right) + \frac{1}{8} - \delta \right\}, \]

where \( dE \) is the energy lost by an electron in travelling a distance \( dx \),

\( N_a \) is Avogadro's number in atoms/gm.-mole,

\( \rho \) is the density of the material in gm/cm\(^3\),

\( Z \) is the nuclear charge number,

\( A \) is the atomic weight in gm/gm.-mole,

\( m \) is the electron rest mass in gm,

\( v \) is the velocity of the incident in cm/sec,

\( E \) is the kinetic energy of the electrons in joules,

\( I \) is the average excitation potential of the atom in joules,

and \( \delta \) is the density correction factor.

In order to simplify subsequent formulae we define a quantity \( n \) by

\[ n = \frac{N_a \rho Z}{A}, \]

where \( n \) is expressed in electrons per cm\(^3\).

The density correction factor \( \delta \) arises when we consider the passage of relativistic particles through matter. The energy loss is reduced because less energy is transferred in distant collisions due to the polarization of the medium by the relativistic electron. The formula for \( \delta \) is from Sternheimer (1952),
where \( p \) is the momentum of the incident electron in gm.cm/sec, 
\( c \) is the velocity of light in cm/sec., 
and \( h \) is Planck's constant in erg.-sec.

\[ \nu_p \] is the plasma frequency defined by

\[ \nu_p = \left( \frac{n \cdot e^2}{\pi} \right)^{\frac{1}{2}} \]

where \( n \) is the classical electron radius = 2.818x10^{-13} cm.,
\( e \) is the electronic charge, in e.s.u.,
and \( n \) is the number of electrons per cm.

Equation (A1.2) is valid in most solids for energies greater than 10 MeV.

With substitution of Equation (A1.2) into (A1.1) we obtain

\[ \frac{dE}{dx} = \frac{2 \pi e^2 n}{\rho mc^2} \beta^2 \left[ \frac{1}{\beta^2} \left( \ln E + \ln \left( \frac{mc^2}{h \nu_p} \right) + 0.43 \right) \right] \]

where \( h \nu_p \), \( E \) and \( mc^2 \) are expressed in MeV. Values of \( h \nu_p \) and \((2\pi e^4 n)/(\rho mc^2)\) for various substances have been tabulated by Sternheimer (1952), (1953) and (1956).

**A1.2 Collision Energy Straggling**

The amount of energy transferred to the atomic electrons during
collision can vary a great deal giving rise to an energy straggling distri-
bution in the transmitted electrons. If the target is thin compared to the
range of the incident electrons, the shape of the straggling distribution
can be calculated theoretically. The equation for the distribution is
given by Birkhoff (1958), p. 87, in terms of a dimensionless parameter
which makes the formula applicable to any material. The approximate
width of the distribution at half height as given by Bethe and Ashkin
(1953), p. 256, is

\[ \delta E = 3.98 \frac{2\pi e^4 n}{mc^2 \rho} \frac{t}{\beta^2}, \]

where \( t \) is the target thickness in gm/cm\(^2\) and \( \delta E \) is in MeV.

Experiments by Goldwasser et al (1952), have found excellent
agreement with the theory, using 15.7 MeV electrons incident on thin
targets of various materials. More recently, Breuer (1964) has measured
the straggling distribution of various target materials using electrons
in the 20 to 60 MeV range. He also found reasonable agreement with
theory.

The curve for the energy straggling distribution is shaped like
a Gaussian except for a long tail as shown in Figure Al.1. The distri-
bution shown here is essentially the curve given by Landau as quoted
by Goldwasser et al (1952).

Rohrlich and Carlson (1954) discuss in some detail the nature of
the energy transfers involved that give rise to the distribution. They
point out that the peak of
the curve is due to frequent
distant collisions in which
the energy transfers are
quite small. On the other
hand, the tail of the distri­
bution is due to relatively
rare, close collisions which
involve a high energy transfer.

Al.3 The Most Probable Energy Loss

The most probable energy loss, defined by the peak of the energy
straggling distribution, differs from the average energy loss because
of the high-energy-transfer tail on the distribution. This is illustrated
in Figure A1.1. Since the annihilation spectrum will be affected mainly
by the most probable energy loss we will use it as the basis of our cal­
culations. For most target thicknesses of interest the energy straggling
distribution can be approximated by a delta function centered on the
most probable energy loss, since the straggling distribution is narrow
compared with width of the annihilation photon spectrum. If however,
the target is thick enough to make the straggling distribution width significant
compared to the width of photon spectrum, we can approximate the stragg­
ling distribution by a Gaussian of half-width $SE$ centered on the most probable energy loss.

Sternheimer (1953) adds a density effect correction to the expression for the most probable energy loss given by Landau. The expression can be written as

$$(Al.6) \quad -\Delta E = \frac{G \ t}{\beta^2} \left\{ \ln t + \ln \left( \frac{Gmc^2}{[h\nu]^2} \right) + 96 \right\},$$

where $G = \frac{2\pi \alpha^4}{mc^2}$ in MeV cm$^2$/g, $t$ is the target thickness penetrated in gm/cm$^2$ such that $|\ln t| < \ln \left( \frac{Gmc^2}{[h\nu]^2} \right) + 96$, and the other symbols have been defined previously in this appendix.

We note that Equation (Al.6) is independent of the incident electron energy, whereas Equation (Al.4) contains the energy in the argument of a logarithm. This is consistent with the fact that the density effect tends to limit the energy transfer to distant collisions as the incident energy increases. Since these numerous distant collisions give rise to the most probable energy loss, we can see why it does not depend on the incident energy. On the other hand, however, the rare close collisions are affected very little by the density effect. This causes the tail of the straggling distribution to become more prominent as the incident energy increases, resulting in an increase in the average energy loss. Thus, instead of the peak of the distribution being shifted
as the energy increases, the distribution changes in shape as the tail becomes more important.

The ionization energy $I$ does not appear in either expression. This is reasonable because a relativistic electron that has an energy much greater than the ionization energy, sees essentially a cloud of free electrons in the target.

**Al. 4 Calculations of Numerical Expressions for $\delta E$ and $\Delta E$**

We can calculate numerical expressions for Equations (Al. 5) and (Al. 6) for a Li–H target. Equation (Al. 5) becomes

$$\delta E = 3.98 \frac{G t}{\beta^2} = 3.04 \times 10^{-1} t,$$

where $t$ is in gm/cm$^2$ and $\delta E$ is in MeV.

Equation (Al. 6) can be written as

$$-\Delta E = 7.63 \times 10^{-2} t [\ln t + 19.40],$$

where $t$ is in gm/cm$^2$ such that

$$|\ln t| < 19.40$$

and $\Delta E$ is in MeV.

**Al. 5 Bremsstrahlung Radiation Energy Losses**

Bremsstrahlung radiation is emitted when an incident electron is
decelerated by the electric field of a nucleus or an atomic electron. The incident electron loses the amount of energy given to the emitted photon. The purpose of this section is to examine the energy losses and straggling distribution of the electrons due to bremsstrahlung. There is some controversy about the differences between the electron bremsstrahlung cross-section as compared to the positron bremsstrahlung cross-section (See Jabbar and Pratt (1963)). The differences between the cross-sections would give rise to different energy loss formulae for each of the particles, but we will assume the differences to be negligible and use the electron bremsstrahlung result in our calculations. The bremsstrahlung process has been studied in detail by many people (see review article by Koch and Motz (1959)).

The problem of calculating the bremsstrahlung cross section becomes quite difficult in the energy range from 10 to 100 MeV because of partial screening of the nucleus by the atomic electrons. At low energies where screening may be neglected, or at high energies where screening is complete, simple analytic expressions may be derived. Screening has the effect of reducing the cross-section and hence the energy loss.

Bethe and Heitler (1934) use the Fermi-Thomas model of the atom to calculate the cross-section. This involves numerical integrations since the form factor for the Fermi-Thomas atom is not given in closed analytic form.

We can compare the average radiation loss with the average col-
lision energy loss. Bethe and Ashkin (1953), p. 206 give an approximate expression for the ratio of the losses:

\begin{equation}
\frac{(dE/dx)_{\text{rad}}}{(dE/dx)_{\text{coll}}} = \frac{E_0 Z}{1600 m^2 c^2},
\end{equation}

where $E_0$ is the incident electron energy and $Z$ is the nuclear charge number.

For 30 MeV electron incident on a LiH target ($Z=4$) we find that the ratio of the average energy losses as given by (AI. 9) is 15 per cent.

We thus see that the average radiation energy loss is important in comparison with the average collision energy loss. These calculations are rough, but they do indicate that the effect must be taken into consideration. It should be noted that the energy at which the ratio expressed by Equation (AI. 9) is 1, is 200 MeV.

Al. 6 Radiation Energy Straggling

In order to calculate the radiation energy straggling of electrons, we will use Heitler's results. He uses an approximate equation for the bremsstrahlung cross section to obtain the following expression (Heitler (1954), p. 378):

\begin{equation}
\omega(y) dy = \frac{e^{-y} y^{b+1}}{\Gamma(b+1)} dy,
\end{equation}

where $b = 0.15 NZ^2 r_0^2$ in which

$N$ is the number of atoms per cm$^3$,  

\[ N \]
\( r \) is the classical electron radius in cm

and \( Z \) is the atomic number;

\( \ell \) is the target thickness in cm,

\( \Gamma(\beta \ell) \) is the gamma function (see, for example, Jahnke et al (1960), p. 4),

and \( \psi = \ln \left( \frac{E_e}{(E_e - h)} \right) \), where \( E_e \) is the initial positron energy and \( h \) is the emitted photon energy.

Equation (A1.10) represents the probability that the energy of an incident electron has been reduced to \( e^{-\psi} \) times its initial value after traveling a distance \( \ell \) in the target.

Since \( \omega(\psi)d\psi \) is the probability that the energy of an electron has been reduced after travelling through a portion \( \ell \) of the target, it represents the energy distribution of the electrons due to radiation straggling. Equation (A1.10) may be expressed in terms of

\[
E = E_e e^{-\psi}
\]

which is a more convenient form for calculations, since this is the energy the electron would have at a depth \( \ell \) in the target. We obtain

\[
(\text{A1.11}) \quad \omega(E) dE = \frac{1}{E_e} \left[ \ln \left( \frac{E_e}{E} \right) \right]^{b \ell - 1} \frac{1}{\Gamma'(b \ell)} \frac{E}{b \ell} dE,
\]

where the substitution \( \psi = \ln \frac{E_e}{E} \) has been made in (A1.10).

For purposes of calculation we consider a LiH target 0.2 cm. in thickness with 30 MeV electrons incident on it. (The actual targets considered in this thesis will be less than 0.2 cm. thick). We find that
\[ b \lambda = 1.41 \times 10^{-3}, \]

so that Equation (Al.11) may be written as

\[ \omega(E) dE = \frac{b \lambda}{E_0} \left[ \ln \left( \frac{E_0}{E} \right) \right] dE, \]

since

\[ n(b \lambda) \approx \frac{1}{b \lambda} \quad \text{and} \quad \left( \ln \frac{E_0}{E} \right)^{b \lambda - 1} \approx \left( \ln \frac{E_0}{E} \right)^{-1}, \]

for \( b \lambda \ll 1 \). Equation (Al.12) is plotted in Figure Al.2 to give the radiation straggling distribution for 30 MeV incident electrons.

![Diagram](image-url)

**Figure Al.2.** Diagram showing the radiation energy straggling distribution for positrons of 30 MeV incident energy after passing through a LiH target 0.2 cm. in thickness.
As can be seen from the figure, the straggling distribution is narrow enough to be considered, for our purposes, a delta function centered at the incident energy. The long tail of the distribution shifts the average energy loss to the left but we see that the most probable energy loss is zero. On the basis of these results we will neglect any energy losses due to bremsstrahlung radiation.
APPENDIX 2
THE COMPUTER PROGRAM

A 2.1 Calculation of \( \lambda_{\infty} \)

In chapter 6 we indicated a method to calculate \( \lambda_{\infty} \). We took \( \lambda_{\infty} \) to be the solution to the equation:

\[
\frac{e^{-\lambda_{\infty}}}{\lambda_{\infty}} I_0(\mu, \lambda_{\infty}) = 1 \times 10^{-5},
\]

where \( 0 \leq \mu \leq 1 \). In order to evaluate \( \lambda_{\infty} \) for a given value of \( \mu \), we can plot

\[
\frac{e^{-\lambda}}{\lambda} I_0(\mu, \lambda)
\]

versus \( \lambda \) on log-log graph paper for various values of \( \mu \). We can then plot \( \mu \) versus log \( \lambda \) for

\[
(A 2.1) \quad \frac{e^{-\lambda}}{\lambda} I(\mu, \lambda) = 1 \times 10^{-5}.
\]

If we put \( \lambda = \lambda_{\infty} \) when Equation (A 2.1) is satisfied, we can obtain the following relationships between \( \mu \) and \( \lambda_{\infty} \):

\[
(A 2.2) \quad \lambda_{\infty, 1} = 8.9 e^{1.08 \mu} \quad (0 \leq \mu \leq 0.6),
\]

\[
(A 2.3) \quad \lambda_{\infty, 2} = 1.45 e^{4.1 \mu} \quad (0.6 \leq \mu \leq 0.9),
\]

and

\[
(A 2.4) \quad \lambda_{\infty, 3} = 1000 e^{28.6(\mu - 1)} \quad (0.9 \leq \mu \leq 1).
\]

The different ranges of \( \mu \) can easily be included in the computer program.
In order to calculate the error involved in choosing $\lambda_\infty$ to be the upper limit of integration, (see Equation (6.19)) rather than infinity, we can check the difference between the integrals

\[(A\ 2.5) \quad R_1(\mu, \lambda) = \int_{\lambda_1}^{\infty} \frac{e^{-3}}{3} I_0(\mu, \zeta) \, d\zeta, \]

and

\[(A\ 2.6) \quad R_2(\mu, \lambda) = \int_{\lambda_1}^{\lambda_\infty} \frac{e^{-3}}{3} I_0(\mu, \zeta) \, d\zeta, \]

for a sufficiently large value of $\lambda_1$. If we note the behaviour of the integrand shown in Figure 6.2 we see that the largest error occurs when $\mu = 1$. In order to calculate the largest value of

$$\lambda_1 = \frac{\Theta_\alpha^2 + \Theta_r^2}{c, \frac{T}{\pi}},$$

we choose $\Theta_r^2 = \Theta_\alpha^2 = 6.25 \times 10^{-4}$. Since

$$\Theta_\alpha^2 = \Theta_r^2$$

for $\mu = 1$ (see Equation (5.18)) we can calculate the value of $k_o - k$. It turns out to be $0.28 \text{ mc}^2$. The value for $c$ may be obtained from the computer results given in the latter part of this appendix. Thus, we calculate the value of $\lambda_1$ to be

$$\lambda_1 \approx 20.$$

In order to evaluate (A 2.5) and (A 2.6) for $\mu = 1$, we can approximate the integrand by
for sufficiently large values of \( \gamma \) (see Equation (6.13)). Thus, the percentage error \( \phi \) that would be involved in choosing an upper limit using Equation (A 2.4) is

\[
\phi = \int_{\gamma_0}^{\infty} - \int_{\gamma_0}^{\infty} \times 100.
\]

Now

\[
\int_{\lambda_1}^{\lambda_2} \frac{1}{\sqrt{2\pi} \gamma} d\gamma = \frac{2}{\sqrt{2\pi}} \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right),
\]

and hence (A 2.7) becomes

\[
\phi = \frac{2}{\sqrt{2\pi}} \left[ \frac{1}{(20)^\frac{1}{2}} - \frac{1}{(20)^\frac{1}{2}} + \frac{1}{(500)^\frac{1}{2}} \right] \times 100 = 14 \%
\]

and so the maximum error is 14 per cent.

We must remember that \( F(\lambda) \) is only one value in the integration over \( \Theta_r \) from which we calculate the spectrum. We have chosen the worst possible case since \( \lambda \) is usually much less than 20 (calculations show that it is seldom greater than 3). The integrand in Equation (A 2.5) approaches zero more rapidly for \( \mu \) less than 1 than it does in the case we have considered and thus less error would be introduced when
is less than 1. We have also seen that this error occurs when

\[ \kappa - \kappa' = 0.28. \]

Consultation of the spectrum for segment 1 in Figure 6, shows that in this case the spectrum has dropped to about 15 per cent of its maximum value. Thus we can say that the error introduced by this approximation is less than the maximum value given by (A 2.8) and probably not more than a few per cent. The reason a larger value of \( \lambda \) was not chosen is because the computer would have taken much longer to perform the integration and it was felt that the extra accuracy involved was not justified. A trial run was made with \( \Omega \) decreased to 7 per cent and the differences were found to be negligible.

A 2.2 Discussion of the Convergence of the Numerical Integration Formula

In Chapter 6, a question was raised concerning the convergence of the integrals to their exact value using numerical integration. Computer results for \( \Delta \) (see Equation (6.18)) equal to .005 and .02 indicate that the integral converges very rapidly.

Berezin and Zhidkov (1963) p. 241 give a formula for the remainder term in Gauss's numerical integration formula:

\[
R = \frac{(\kappa - a)^{2n+1} (n!)^2 f_{max}(\delta)}{(2\pi n) L^2 (2n+1)}, \quad (a \leq \delta \leq b)
\]

where we are considering numerical evaluation of the integral

\[ \int_a^b f(x) \, dx. \]
In Equation (A 2.9) \( n = 12 \), is the number of intervals we have chosen for Gauss's integration formula, and \( \xi \) is chosen such that \( f^{2n}(\xi) \), the 2\(n\)-th derivative of \( f(\xi) \), is a maximum value in the range of integration \((a, b)\). The remainder term (A 2.9) gives the difference between the value of the integral calculated using \( n \)-interval numerical integration and the exact value of the integral. In order to compute the error involved in the integrals given in Chapter 6 we must take the 24\(th\) derivative of their integrands. This would be extremely laborious and will not be done. A possible approach would be to use a value of \( n \), smaller than 12, to calculate (A 2.9). The value of \( R \) obtained in this manner would give an upper limit on the error involved in the integrals considered in this thesis. Rather than do this however, we will assume that the integrals converge rapidly enough so that any error involved is negligible for our purposes.

### A 2.3 The Computer Program

The computer program is written in Fortran II. In order to facilitate debugging, the program is set up using a main program and subprograms. The main program acts as an executive program, whereas the subprograms perform the calculations.

The list of variables used in the program are given in Table A2.1 below.
Table A 2.1

List of Variables used in the computer program

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(M)</td>
<td>Mth root of $P_n(u)$ (Gauss's formula)</td>
</tr>
<tr>
<td>TMIN</td>
<td>minimum value of $t$ for segment 1</td>
</tr>
<tr>
<td>ZMAX</td>
<td>$\lambda \infty$</td>
</tr>
<tr>
<td>RATIO</td>
<td>$\Delta$</td>
</tr>
<tr>
<td>FZMIN</td>
<td>smallest value of $e^{-3/2} I_0(\mu_3) = 1.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>KMAX</td>
<td>number of values of $k_o - k$</td>
</tr>
<tr>
<td>ISEG</td>
<td>number of segments</td>
</tr>
<tr>
<td>THC</td>
<td>$\Theta_c$</td>
</tr>
<tr>
<td>EINC</td>
<td>$U_\tau$</td>
</tr>
<tr>
<td>ESCAT(K)</td>
<td>Kth value of $k_o - k$</td>
</tr>
<tr>
<td>T1</td>
<td>$t_{j-1}$</td>
</tr>
<tr>
<td>T2</td>
<td>$t_j$</td>
</tr>
<tr>
<td>KSEG</td>
<td>jth segment number</td>
</tr>
<tr>
<td>TAV</td>
<td>$\overline{\tau}_j$</td>
</tr>
<tr>
<td>ESEG</td>
<td>$U_\tau - 2.05 \overline{\tau}_i$</td>
</tr>
<tr>
<td>CEE</td>
<td>$\overline{\tau}_j$</td>
</tr>
<tr>
<td>EKO</td>
<td>$k_o(t)$</td>
</tr>
<tr>
<td>EPSI</td>
<td>$k_o - k/k_o$</td>
</tr>
<tr>
<td>THA2</td>
<td>$\Theta_a^2$</td>
</tr>
<tr>
<td>THA</td>
<td>$\Theta_a$</td>
</tr>
<tr>
<td>DSDK</td>
<td>$\frac{2}{\overline{\tau}_i} \cdot \frac{11503}{k \left[1 - 2(k_o - k)\right]}$</td>
</tr>
</tbody>
</table>
Table A 2.1 (Continued)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>THG2</td>
<td>$\Theta_Y$</td>
</tr>
<tr>
<td>THG</td>
<td>$\Theta_Y$</td>
</tr>
<tr>
<td>VEE</td>
<td>$\Theta_Y + \Theta_v$</td>
</tr>
<tr>
<td>GMU</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Z1</td>
<td>$\lambda_{j-1}$</td>
</tr>
<tr>
<td>Z2</td>
<td>$\lambda_j$</td>
</tr>
<tr>
<td>FUZ</td>
<td>$(e^{-2/3}) I_0(\mu z)$</td>
</tr>
<tr>
<td>FULAM</td>
<td>$F(\mu \lambda)$</td>
</tr>
<tr>
<td>FTHG</td>
<td>$F(\mu \lambda) \Theta_Y$</td>
</tr>
<tr>
<td>QUAD</td>
<td>$\int F(\mu \lambda) \Theta_Y d\Theta_Y$</td>
</tr>
<tr>
<td>THINT</td>
<td>$dY/dk$</td>
</tr>
<tr>
<td>ZNEG</td>
<td>$\lambda_{j-1} - \lambda_j$</td>
</tr>
<tr>
<td>ZEDI</td>
<td>$\lambda_{j-1}$ in subroutine Gauss</td>
</tr>
<tr>
<td>ZED2</td>
<td>$\lambda_j$ in subroutine Gauss</td>
</tr>
</tbody>
</table>

In this computer program all target thickness are changed from mm. to radiation lengths (See Table 2.2) by dividing the thickness in mm. by 922.0.

The computer program is given on the following pages:
COMPUTER PROGRAM TO CALCULATE THE PHOTON SPECTRUM

MAIN PROGRAM

C POSITRON ANNIHILATION MAIN PROGRAM

DIMENSION ESCAT(30),U(12)
COMMON U,MOT,TMIN,ZMAX,RATIO,FZMIN,ESCAT,KMAX,THC,EINC,T1,T2,
1KSEG,THA,THA2,CES,DSDK,THINT,ISEG
U(1)=62666704E-1
U(2)=U(1)
U(3)=-18391575
U(4)=-U(3)
U(5)=.29365898
U(6)=-U(5)
U(7)=-.38495134
U(8)=-U(7)
U(9)=.45205863
U(10)=-U(9)
U(11)=.49078032
U(12)=-U(11)
CALL READB
CALL PRINTA(ISEG,KMAX,EINC,THC)
EINC=EINC/510976
DO 93 MSEG =1,ISEG
CALL READC
TAV=(T2+T1)/2.
ESEG=EINC-2.05*TAV
CEE=(1073.0+90.7*LOGF(TAV/92.1))/ESEG**2
EKO=ESEG**.5
CALL PRINTB(KSEG,CEE,EKO)
IF (THINT.100.93.100 CONTINUE
12 T1=TMIN
10 CONTINUE
DO 100 K=1,KMAX
EPSI=ESCAT(K)/EKO
THA2=2.*EPSI/EKO
THA=SRTF(THA2)
DSDK=22.6/(1+2.*EPSI*EKO)*EKO*CEE)
CALL INTEGA
CALL PRINTC(ESCAT(K),THINT,DSDK,EKO)
IF (THINT.100.93.100 CONTINUE
93 CONTINUE
STOP
END

SUBPROGRAMS

SUBROUTINE READB
DIMENSION ESCAT(30),U(12)
COMMON U,MOT,TMIN,ZMAX,RATIO,FZMIN,ESCAT,KMAX,THC,EINC,T1,T2,
1KSEG,THA,THA2,CES,DSDK,THINT,ISEG
READ1.TMIN.ZMAX.RATIO.FZMIN
1 FORMAT(4E10.4)
READ 2.KMAX.ISEG.THCEINC.(ESCAT(J)+J=1.KMAX)
2 FORMAT(213.2E14.8/(E10.4))
RETURN
END

SUBROUTINE PRINTA(ISEG,KMAX.EINC.TH)
THING=EINC*510976
PUNCH4
4 FORMAT(/10X,38HPOSITRON ANNIHILATION IN THICK TARGETS//)
PUNCH5.ISEG.KMAX.EINC.THGC
5 FORMAT(12HNO OF SEG -.12.18H NO 0' VAL
2 C =-.E14.7//)
RETURN
END

SUBROUTINE READC
DIMENSION ESCAT(30),U(12)
COMMON U,MDAT.TMIN.ZMAX.RATIO.FZMIN.ESCAT,KMAX.THCEINC.T1.T2.
1KSEG,THA.THA2,CEE.DALK.THINT.ISEG
READ 3.T1.T2.KSEG
3 FORMAT(2E10.4.12)
RETURN
END

SUBROUTINE PRINTB(KSEG.CEE.EKO)
PUNCH5.KSEG.CEE.EKO
5 FORMAT(2X.17H SEGMENT NUMBER
CEE -.£14.21H. MAX
ENERGY -.E14.7//)
RETURN
END

SUBROUTINE INTEGA
DIMENSION ESCAT(30),U(12),FTHG(12)
COMMON U,MDAT.TMIN.ZMAX.RATIO.FZMIN.ESCAT.KMAX.THCEINC.T1.T2.
1KSEG,THA.THA2,CEE.DALK.THINT.ISEG
INN = 1
LIV=1
PREV=0.0
TH1=0.0
FRED=0.0
FLOAT=LIV
XNC =THC/FLOAT
DO 20 I=1,LIV
TH2=TH1+ XNC
DO 40 J=1,12
THG = XNC*U(J)+(TH1+TH2)/2.0
THG2 = THG**2
VEE = THA2 + THG2
GMU = 2 * THA * THG / VEE
Z1 = VEE * 92.1 / (CEE * T2)
Z2 = VEE * 92.1 / (CEE * T1)
FZ = FUZ(GMU, Z1)
IF (FZ - FZMIN) 42, 42, 41
41 IF (GMU < 6) 4, 5, 5
4 ZMAX = 8.9 * EXPF(1.08 * GMU)
19 GOTO 73
73 IF (Z2 - ZMAX) 43, 43, 44
5 IF (GMU > 9) 6, 7, 7
6 ZMAX = 1.45 * EXPF(4.1 * GMU)
GOTO 19
7 ZMAX = 1000 * EXPF(28.6 * (GMU - 1.0))
GO TO 19
44 Z2 = ZMAX
IF (Z2 - Z1) 49, 49, 43
49 GO TO 42
43 FUL = GAUSS(Z1, Z2, GMU)
FTHG(J) = FUL * THG
GO TO 40
42 FTHG(J) = 0.0
40 CONTINUE
QUAD = XNC * (1.12457352 * (FTHG(1) + FTHG(2)) + 1.1674627 * (FTHG(3) + FTHG(4)) + 1.0158371 * (FTHG(5) + FTHG(6)) + 8039165E-1 * (FTHG(7) + FTHG(8)) + 2.53969663E-1 * (FTHG(9) + FTHG(10)) + 2.3587668E-1 * (FTHG(11) + FTHG(12)))
FRED = FRED + QUAD
IF (QUAD) 20, 62, 20
62 GO TO 20
20 TH1 = TH2
IF (FRED) 51, 10, 51
51 IF (SENSE SWITCH 1) 10, 10, 12
18 IF (ABSF(PREV - FRED) / FRED - RATIO) 10, 10, 12
12 PREV = FRED
LIV = 2 * LIV
INN = INN + 1
GO TO 92
10 THINT = DSDK * FRED
RETURN
END

SUBROUTINE PRINTC(ESCAN, THINT, DSDK, EKO)
Y = EKO - ESCAN
T = Y * 510976
DAMN = THINT / 510976
PUNCH6, ESCAN, DSDK, T, DAMN, Y, THINT
RETURN
END
FUNCTION FUZ(R*Z)
U=R*Z
IF(U-2.5)50.50.51
T=1./U
G=(1.+T*(1./U+9./128.*T))
FUZ=EXP(-U)*G/(SQRT(U+2.*3.*141593)*Z)
RETURN
50 G=(U/Z.)**2
B=1.+Q*(1.+Q*(-25+1./36.*Q))
FUZ=EXP(-Z)*B/Z
RETURN
END

FUNCTION GAUSS(Z1,Z2,GMU)
DIMENSION ESCAT(30),U(12)
COMMON U,MDAT,TMIN,ZMAX,RATIO,FZMIN,ESCAT,KMAX,THC,EINC,T1,T2,
KSEG,THA,THA2,CEE,DSDK,THINT,ISEG
INN=1
IDIV=1
PREV=0.0
ZNEG=Z2-Z1
5 ZED1=Z1
GAUSS=0.0
FLOAT=IDIV
XINVAL=ZNEG/FLOAT
DO 20 J=1,IDIV
ZED2=ZED1+XINVAL
FLAM=FULAM(ZED1,ZED2,GMU,U)
GAUSS=GAUSS+FLAM
20 ZED1=ZED1+XINVAL
IF(SENSE SWITCH 2)40,1
1 IF(ABS(PREV-GAUSS)/GAUSS-RATIO)40,30
30 PREV=GAUSS
IDIV=IDIV*2
INN=INN+1
GO TO 5
40 GO TO 37
37 RETURN
END

FUNCTION FULAM(Z1,Z2,GMU,U)
DIMENSION U(12),FUZ(12)
ZPOS=(Z1+Z2)/2.
ZNEG=Z2-Z1
DO 80 M=1,12
Z=ZPOS+ZNEG*U(M)
FUZA(M)=FUZ(GMU,Z)
80 CONTINUE
FULAM=ZNEG*(-1.24516532*(FUZA(1)+FUZA(2))+116746.27*(FUZA(3)+FUZA(1))
+1015831.7*(FUZA(5)+FUZA(6))+80039165.E-1*(FUZA(7)+FUZA(8))
+2.54369663.E-1*(FUZA(9)+FUZA(10))+23587668.E-1*(FUZA(11)+FUZA(12))
RETURN
END
A 2.4 Input Data

The data used in the calculations at 15 MeV are given on the following page. In this case $\Delta$ has been set at $\Delta \%$. A provision has been made in the computer to switch out $\Delta$ so that each integral can be calculated for 12 intervals only. If $\Delta$ is not switched out the computer will calculate the integrals for an increasing number of segments until the inequality given by Equation (6.18) is satisfied. In order to switch $\Delta$ out for the integral over $z$, program switch 2 must be put in the "on" position. In order to switch $\Delta$ out for the integral over $\Theta$, program switch 1 must be put in the "on" position. Either of these switches can be used at any time in the course of the calculations, but care must be taken if the results are to be accurate.

In the results given in this thesis both switches were left off for calculations in the first segment. In other segments switch 2 was left off for $\Theta = 0$ but was turned on for the rest of the calculations.
INPUT DATA FOR PROGRAM

8 SEGMENTS OF 0.2MM THICKNESS - 1 PER CENT ERROR

CARDS FOR READA

1.000E-5  0.0000E+0  1.000E-1  4.0000E-4
0.0000E+0
4.0000E-1
1.0000E+0
2.0000E+0
3.0000E+0
4.5000E+0
6.0000E+0
8.0000E+0
1.0000E+1
1.4000E+1
2.0000E+1

CARDS FOR READB

0.0000E+0  2.0000E-1  1
2.0000E-1  4.0000E-1  2
4.0000E-1  6.0000E-1  3
6.0000E-1  8.0000E-1  4
8.0000E-1  1.0000E+0  5
1.0000E+0  1.2000E+0  6
1.2000E+0  1.4000E+0  7
1.4000E+0  1.6000E+0  8
A 2.5 Results from Computer Calculation

When the data given in Section A 2.4 is used in the computer program given in Section A 2.3 the results given in the next 7 pages are obtained. These results are for a target 1.6 mm. in thickness that has been divided into 8 segments. The incident positron energy is 15 MeV, the photon acceptance angle is $2.5 \times 10^{-2}$ radians and $\Delta$ has been set at 1 per cent.

Most of the variables given are self-explanatory, or have been given previously; however the following table gives the variables not already defined.

<table>
<thead>
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<td>Variables that appear in computer output</td>
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<table>
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<th>ABS EN (MEV)</th>
<th>$\kappa_1$ in MeV</th>
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<tr>
<td>$\lambda / E+/\text{MEV}$</td>
<td>$dy/d\kappa$ in photons/positron/MeV</td>
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<tr>
<td>$\lambda / E+/\text{MOC}$</td>
<td>$dy/d\kappa$ in photons/positron/$\text{mc}^2$</td>
</tr>
<tr>
<td>KO-K</td>
<td>$\kappa_0 - \kappa$ in $\text{mc}^2$ units</td>
</tr>
<tr>
<td>THETA C</td>
<td>$\theta_c$</td>
</tr>
</tbody>
</table>
POSITRON ANNIHILATION IN THICK TARGETS

NO OF SEG = 8  NO OF VAL KO-K = 11  INCIDENT ENERGY (MEV) = 1.5000000E+01  INCIDENT ENERGY (MOC) = 2.9355586E+01
THETA C = 2.5000000E-02

SEGMENT NUMBER = 1
CEE = 2.8480294E-01  MAX PHOTON ENERGY = 2.9835086E+01

KO-K = 0.0000000E-99  DSDK = 2.6597246E+00  ABS EN (MEV) = 1.5245012E+01  Y/E+/MEV = 1.6119326E-04  ABS EN (MOC) = 2.9835086E+01  Y/E+/MOC = 8.2365891E-05

KO-K = 4.0000000E-02  DSDK = 2.4627083E+00  ABS EN (MEV) = 1.5224573E+01  Y/E+/MEV = 1.4508062E-04  ABS EN (MOC) = 2.9795086E+01  Y/E+/MOC = 7.4132716E-05

KO-K = 1.0000000E-01  DSDK = 2.2164374E+00  ABS EN (MEV) = 1.5193915E+01  Y/E+/MEV = 1.3035299E-04  ABS EN (MOC) = 2.9735086E+01  Y/E+/MOC = 6.6607046E-05

KO-K = 2.0000000E-01  DSDK = 1.8998035E+00  ABS EN (MEV) = 1.5142817E+01  Y/E+/MEV = 9.3694312E-05  ABS EN (MOC) = 2.9635086E+01  Y/E+/MOC = 4.7875545E-05

KO-K = 3.0000000E-01  DSDK = 1.6623280E+00  ABS EN (MEV) = 1.5091720E+01  Y/E+/MEV = 3.5036977E-05  ABS EN (MOC) = 2.9535086E+01  Y/E+/MOC = 1.7903024E-05

KO-K = 4.5000000E-01  DSDK = 1.3998551E+00  ABS EN (MEV) = 1.5015073E+01  Y/E+/MEV = 3.2584731E-06  ABS EN (MOC) = 2.9385086E+01  Y/E+/MOC = 1.6650016E-06

KO-K = 6.0000000E-01  DSDK = 1.2089658E+00  ABS EN (MEV) = 1.4938427E+01  Y/E+/MEV = 2.2446897E-07  ABS EN (MOC) = 2.9235086E+01  Y/E+/MOC = 1.4698262E-07

KO-K = 8.0000000E-01  DSDK = 1.0229710E+00  ABS EN (MEV) = 1.4836232E+01  Y/E+/MEV = 0.0000000E-99  ABS EN (MOC) = 2.9035086E+01  Y/E+/MOC = 0.0000000E-99

SEGMENT NUMBER = 2
CEE = 4.0171662E-01  MAX PHOTON ENERGY = 2.9794086E+01

KO-K = 0.0000000E-99  DSDK = 1.8882460E+00  ABS EN (MEV) = 1.5224062E+01  Y/E+/MEV = 1.5954939E-04  ABS EN (MOC) = 2.9794086E+01  Y/E+/MOC = 8.1525912E-05

KO-K = 4.0000000E-02  DSDK = 1.7483761E+00  ABS EN (MEV) = 1.5203623E+01  Y/E+/MEV = 1.4154559E-04  ABS EN (MOC) = 2.9754086E+01  Y/E+/MOC = 7.236402E-05
KO-K = 1.0000000E-01 DSDK = 1.5735384E+00 ABS EN (MEV) = 1.5172965E+01 Y/E+/MEV = 1.1378516E-04 ABS EN (MOC) = 2.9694086E+01 Y/E+/MOC = 5.8141488E-05

KO-K = 2.0000000E-01 DSDK = 1.3487473E+00 ABS EN (MEV) = 1.5121867E+01 Y/E+/MEV = 7.0893022E-05 ABS EN (MOC) = 2.9594086E+01 Y/E+/MOC = 3.6224623E-05

KO-K = 3.0000000E-01 DSDK = 1.1801538E+00 ABS EN (MEV) = 1.5070770E+01 Y/E+/MEV = 3.9073909E-05 ABS EN (MOC) = 2.9494086E+01 Y/E+/MOC = 1.9965830E-05

KO-K = 4.5000000E-01 DSDK = 9.9381377E-01 ABS EN (MEV) = 1.4994123E+00 Y/E+/MEV = 4.0594055E-06 ABS EN (MOC) = 2.9344086E+00 Y/E+/MOC = 6.8895590E-06


KO-K = 8.0000000E-01 DSDK = 7.2624850E-01 ABS EN (MEV) = 1.4815282E+00 Y/E+/MEV = 7.2318979E-07 ABS EN (MOC) = 2.8994086E+01 Y/E+/MOC = 2.0742588E-06

KO-K = 1.0000000E-01 DSDK = 6.2941535E-01 ABS EN (MEV) = 1.4713086E+00 Y/E+/MEV = 1.749809E-07 ABS EN (MOC) = 2.8794086E+01 Y/E+/MOC = 6.0038709E-08

KO-K = 1.4000000E-00 DSDK = 4.9690687E-01 ABS EN (MEV) = 1.4508696E+00 Y/E+/MEV = 0.0000000E-99 ABS EN (MOC) = 2.8394086E+01 Y/E+/MOC = 0.0000000E-99

SEGMENT NUMBER = 3
CEE = 4.5698577E-01 MAX PHOTON ENERGY = 2.9753086E+01

KO-K = 0.0000000E-99 DSDK = 1.6621636E+00 ABS EN (MEV) = 1.5203112E+01 Y/E+/MEV = 1.4834912E-04 ABS EN (MOC) = 2.9753086E+01 Y/E+/MOC = 7.5802841E-05

KO-K = 4.0000000E-02 DSDK = 1.5390404E+00 ABS EN (MEV) = 1.5182673E+01 Y/E+/MEV = 1.2606192E-04 ABS EN (MOC) = 2.9713086E+01 Y/E+/MOC = 6.4414619E-05

KO-K = 1.0000000E-01 DSDK = 1.3851364E+00 ABS EN (MEV) = 1.5152015E+01 Y/E+/MEV = 9.7291561E-05 ABS EN (MOC) = 2.9653086E+01 Y/E+/MOC = 4.9713653E-05

KO-K = 2.0000000E-01 DSDK = 1.1872597E+00 ABS EN (MEV) = 1.5109917E+01 Y/E+/MEV = 6.1476979E-05 ABS EN (MOC) = 2.9553086E+01 Y/E+/MOC = 3.1413261E-05

KO-K = 3.0000000E-01 DSDK = 1.0385222E+00 ABS EN (MEV) = 1.5049820E+01 Y/E+/MEV = 3.7770934E-05 ABS EN (MOC) =
2.9453086E+01 Y/E+/MOC = 1.9300041E-05

KO-K = 4.5000000E-01 DSDK = 8.7482297E-01 ABS EN (MEV) =
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KO-K = 6.0000000E-01 DSDK = 7.5552892E-01 ABS EN (MEV) =
1.486527E+01 Y/E+/MEV = 7.7365375E-06 ABS EN (MOC) =
2.9153086E+01 Y/E+/MOC = 3.9531850E-06

KO-K = 8.0000000E-01 DSDK = 6.3929369E-01 ABS EN (MEV) =
1.4794332E+01 Y/E+/MEV = 2.5072095E-06 ABS EN (MOC) =
2.8953086E+01 Y/E+/MOC = 1.2811239E-06

KO-K = 1.0000000E+00 DSDK = 5.5405452E-01 ABS EN (MEV) =
1.4692136E+01 Y/E+/MEV = 7.8408132E-07 ABS EN (MOC) =
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KO-K = 4.0000000E+00 DSDK = 4.3741147E-01 ABS EN (MEV) =
1.4487746E+01 Y/E+/MEV = 7.0343595E-08 ABS EN (MOC) =
2.8353086E+01 Y/E+/MOC = 3.5943889E-08

KO-K = 2.0000000E+00 DSDK = 3.3243272E-01 ABS EN (MEV) =
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2.7753086E+01 Y/E+/MOC = 0.0000000E-99

SEGMENT NUMBER = 4
CEE = 4.9403222E-01 MAX PHOTON ENERGY = 2.9712086E+01

KO-K = 0.0000000E-99 DSDK = 1.5396430E+00 ABS EN (MEV) =
1.5182162E+01 Y/E+/MEV = 1.3115912E-04 ABS EN (MOC) =
2.9712086E+01 Y/E+/MOC = 6.7019166E-05

KO-K = 4.0000000E-02 DSDK = 1.4255955E+00 ABS EN (MEV) =
1.5161723E+01 Y/E+/MEV = 1.1035036E-04 ABS EN (MOC) =
2.9672086E+01 Y/E+/MOC = 5.6386386E-05

KO-K = 1.0000000E-01 DSDK = 1.2830359E+00 ABS EN (MEV) =
1.5131065E+01 Y/E+/MEV = 8.5267828E-05 ABS EN (MOC) =
2.9612086E+01 Y/E+/MOC = 4.3569814E-05

KO-K = 2.0000000E-01 DSDK = 1.0997450E+00 ABS EN (MEV) =
1.5079967E+01 Y/E+/MEV = 5.5488269E-05 ABS EN (MOC) =
2.9512086E+01 Y/E+/MOC = 2.8353174E-05

KO-K = 3.0000000E-01 DSDK = 9.6227695E-01 ABS EN (MEV) =
1.5028870E+01 Y/E+/MEV = 3.6138002E-05 ABS EN (MOC) =
2.9412086E+01 Y/E+/MOC = 1.8465652E-05

KO-K = 4.0000000E-01 DSDK = 8.1033845E-01 ABS EN (MEV) =
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KO-K = 6.0000000E-01 DSDK = 6.9983775E-01 ABS EN (MEV) =
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APPENDIX 3

A3.1 Evaluation of Most Probable Scattering Angle

We wish to calculate the most probable scattering angle for the scattering distribution given by Equation (4.19a):

\[ P(\theta_s, t) \, d\theta_s = e^{-\frac{\theta_s^2}{c t}} \, \frac{2 \theta_s}{c t} \, d\theta_s. \]  

This equation represents the probability of a positron being multiply scattered into \( d\theta_s \) at \( \theta_s \) in any azimuthal direction after having penetrated the target to a depth \( t \). The probability for scattering per unit angle is

\[ P(\theta_s, t) = e^{-\frac{\theta_s^2}{c t}} \, \frac{2 \theta_s}{c t}. \]

The most probable scattering angle can be calculated from (A3.2) by differentiating \( P(\theta_s, t) \) with respect to \( \theta_s \) and setting the result equal to zero:

\[ \frac{\partial P(\theta_s, t)}{\partial \theta_s} = -2 \frac{\theta_s}{c t} e^{-\frac{\theta_s^2}{c t}} \frac{2 \theta_s}{c t} + \frac{2}{c t} e^{-\frac{\theta_s^2}{c t}} = 0, \]

and thus

\[ \theta_s^2 = \frac{c t}{2} \quad \text{or} \quad \theta_s = \sqrt{\frac{c t}{2}}, \]

where \( \sqrt{ct/2} \) is the most probable scattering angle.
A3.2 Evaluation of $\int_{0}^{r} e^{-\Theta_t/ct} \, dt$

We wish to evaluate the integral

(A3.4) \[ I = \int_{0}^{r} e^{-\Theta_t/ct} \, dt \] .

If we make the substitutions

(A3.5) \[ u = \frac{l}{t} \quad \text{and} \quad g = \frac{l}{T} \] ,

we obtain the transformed integral:

(A3.6) \[ I = \int_{\infty}^{g} e^{-\Theta_t/ct} \, d\left(\frac{l}{u}\right) . \]

(A3.6) can be integrated by parts to yield

(A3.7) \[ I = \frac{l}{g} e^{-\Theta_t^2/g} + \frac{\Theta_t^2}{c} \int_{g}^{\infty} \frac{l}{\Theta_t^2 u} e^{-\Theta_t^2/ct} \, d\left(\frac{\Theta_t^2 u}{c}\right) . \]

Now, from Jahnke and Emde (1945), p. 1, the exponential integral is defined as

(A3.8) \[ -E_i(-x) = \int_{x}^{\infty} \frac{e^{-t}}{t} \, dt , \]

where \(-E_i(-x)\) is tabulated on p. p. 7, 8 of the above reference, for \(x=0\) to 15. Thus Equation A3.7 can be put in the form

(A3.7a) \[ I = \frac{l}{g} e^{-\Theta_t^2/g} - \frac{\Theta_t^2}{c} E_i\left(\frac{-\Theta_t^2}{c}\right) . \]

Since \(g = 1/T\) , we may write this equation as

(A3.9) \[ I = T e^{-\Theta_t^2/cT} - \frac{\Theta_t^2}{c} E_i\left(\frac{-\Theta_t^2}{cT}\right) , \]

which is the result used in Chapter 7.
For $\Theta_s^2/cT \gg 1$ we get the asymptotic representation (Jahnke et al (1960), p. 18):

$$-E_i\left(-\frac{\Theta_s^2}{cT}\right) \simeq \frac{cT}{\Theta_s^2} e^{-\frac{\Theta_s^2}{cT}},$$

and thus (A3.9) can be written as

(A3.10) \[ I = 2T e^{-\frac{\Theta_s^2}{cT}}, \]

a result that is used in Chapter 7.
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